

DC power grids with constant-power loads—Part I: A full characterization of power flow feasibility, long-term voltage stability and their correspondence

Mark Jeeninga*, Claudio De Persis[†], Arjan van der Schaft[†]

Abstract—In this two-part paper we develop a unifying framework for the analysis of the feasibility of the power flow equations for DC power grids with constant-power loads.

In Part I of this paper we present a detailed introduction to the problem of power flow feasibility of such power grids, and the associated problem of selecting a desirable operating point which satisfies the power flow equations. We introduce and identify all long-term voltage semi-stable operating points, and show that there exists a one-to-one correspondence between such operating points and the constant power demands for which the power flow equations are feasible. Such operating points can be found by solving an initial value problem, and a parametrization of these operating points is obtained. In addition, we give a full characterization of the set of all feasible power demands, and give a novel proof for the convexity of this set.

Index Terms—Power flow analysis, DC power grids, constant-power loads, voltage stability

I. INTRODUCTION

A classical problem in the study of power grid stability is the long-term voltage stability problem. The problem concerns the long-term (in)stability of a power grid due to limitations in the transportation of power from sources to loads. These limits in power transportation are due to a combination of generation limits, load limits and/or limits due to the network (infra)structure. The transportation of power from sources to loads is known as *power flow* or *load flow*, and is captured in the *power flow equations* (also called *load flow equations*). Consequently, the power flow equations implicitly describe the limitations of power flow in a power grid.

Another motivation for the study of power flow are phenomena such as voltage drop, voltage collapse and power outages. Such phenomena may occur when transportation limits are exceeded and the power flow equations cannot be satisfied, in which case no equilibrium of the system can exist. Naturally the existence of equilibria is crucial for the control of power grids. Loosely speaking, control schemes which are designed to satisfy the power flow equations in the long-term time scale may display unintended behavior in the absence of equilibria. A possible consequence is that critical components may reach their operational limits, start to fail, and cause a chain reaction

of more failures. Satisfaction of the power flow equations is therefore crucial to guarantee (long-term) safe operation of the power grid.

Long-term voltage instability is a load-driven phenomenon [1], and different load characteristics may be considered for analysis—see, e.g., [2]. For a load characteristic with given parameters, we refer to solutions of the power flow equations as *operating points* of the power grid. The power flow equations are *feasible* if at least one operating point exists. In general, the power flow equations are nonlinear, and no operating points may exist. Likewise, multiple operating points may exist, while a single operating point should be selected.

For practical power grids there are several distinct properties to select an operating point. First, it is desirable that an operating point is *long-term voltage stable*, meaning that locally all voltage magnitudes of an operating point decrease if any load demand increases. This is to say that the Jacobian of the voltage magnitudes at the loads as a function of the power demand has negative entries [3]. Second, it is desirable that the selected operating point is the solution to the power flow equations that minimizes the total power dissipated in the lines at steady state. A third property is that the operating point is a *high-voltage solution*, meaning that the selected operating point element-wise dominates all other operating point that satisfy the power flow equation.

It is *a priori* not clear if, or under which conditions, these types of operating points coincide, or give rise to a unique operating point. For specific types of power grids, like the power grids that are considered in the current paper, it has been shown that these types of operating point coincide “almost surely”, and that a sufficient condition exists for the uniqueness of the long-term voltage stable operating point [4]. A similar result for a general power grid is not available in the literature, but several sufficient conditions for are known [5], [3], [6].

There have been several publications on the problem of long-term voltage stability, and in particular on the feasibility of power flow equations. Here we list a few of them.

The paper [6] considers a generic AC power grid with a single source, whereas the paper [3] considers a lossless AC power grid. Both [3] and [6] give a sufficient condition for feasibility of the reactive power flow equations, and use fixed-point methods to conclude the existence of an operating point. Estimates of the operating point are also given. In addition, [3] shows that this operating point is the high-voltage solution, and that it is long-term voltage stable.

The paper [7] considers a general power transportation system

*Politecnico di Torino, 10129 Torino, Italy (e-mail: mark.jeeninga@polito.it)

[†]University of Groningen, 9747AG Groningen, The Netherlands (e-mail: {c.de.persis, a.j.van.der.schaft}@rug.nl)

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at steady state and proves a necessary conditions for feasibility of a power demand, which is also sufficient in certain cases. The paper [4] presents an algorithm to determine if the power flow equations of a DC power grid are feasible, and shows that there exists a high-voltage solution, which is “almost surely” long-term voltage stable. Conditions for the long-term voltage stable operating point to be unique are given.

The paper [8] proves that the set of feasible solutions to the power flow equations for DC power grids is convex, as follows from the study of convexity of the joint numerical range of a generalized quadratic form.

In this two-part paper we focus on the power flow of DC power grids with constant-power loads, and assume there are no limits on voltage potentials and line currents. While many types of power grids are studied in the literature, DC power grids with constant-power loads are among the simplest type of power grid where feasibility of the power flow equations is nontrivial.

It is noted that there are several types of power grids for which the power flow equations are equivalent to or well-approximated by the power flow of DC power grids with constant-power loads. The papers [9], [3] show a how the reactive power flow problem for a lossless AC power grid may be approximated by a DC power flow grid model. See also [6], [4] for other examples of power grids with constant-power components.

Even though the literature provides handles to study power flow, the interplay between the different results is not clear, and an over-arching analysis is missing. The main motivation of this paper is to bridge these gaps in the literature for DC power grids, and to develop a unified framework for the analysis of DC power flow with constant-power loads.

Contribution

This paper is split into two parts. Part I of this paper presents a geometric framework to analyze the feasibility of the DC power flow equations with constant-power loads. This framework is extended in Part II to unify and generalize the main contributions of the previously mentioned publications in the context of DC power flow. The novelty of our approach is that we combine results in matrix theory, convex analysis, and initial value problems to analyze DC power flow. In contrast to other approaches, we do not explicitly rely on fixed point analysis or iterative methods.

The main objective of this twin paper is to analyze the set, denoted by \mathcal{F} , of constant power demands for which the power flow equations are feasible. We would like to emphasize that these constant power demands are not sign-restricted. *I.e.*, the power demand at a load is allowed to be negative, in which case the load provides power to the grid. We let \mathcal{D} denote the set of long-term voltage stable operating points. We refer to the vectors in $\text{cl}(\mathcal{D})$, the closure of \mathcal{D} , as *long-term voltage semi-stable* operating points.

In Part I of this paper we develop a geometric framework for DC power flow feasibility for constant-power loads. The main contributions of Part I are as follows.

M1. We give a parametrization of \mathcal{D} , its closure and its boundary, which establishes a constructive method to

describe the long-term voltage (semi-)stable operating points (Theorem 3.7).

M2. For each vector of power demand that lies on the boundary of \mathcal{F} there exists a unique corresponding operating point which solves the power flow equations. Moreover, these operating points form the boundary of \mathcal{D} (Corollary 3.20).

M3. There is a one-to-one correspondence between the feasible power demands \mathcal{F} and the long-term voltage semi-stable operating points $\text{cl}(\mathcal{D})$. This means that if the power flow equations are feasible, then there exists a unique long-term voltage semi-stable operating point that solves the power flow equations. This operating point can be found by solving an initial value problem (Theorem 3.17).

M4. We give a novel and insightful proof for the fact that the set \mathcal{F} is closed and convex. Consequently, \mathcal{F} is the intersection of all supporting half-spaces of \mathcal{F} . We describe all such half-spaces, which gives a complete geometric characterization of \mathcal{F} (Theorem 3.18).

Part II of this paper continues the approach, and recovers and extends several results of the previously mentioned publications for DC power grids with constant-power loads. The main contributions of Part II are as follows.

M5. We prove **(a)** a necessary and sufficient LMI condition for the feasibility of the power flow equations, and a necessary and sufficient LMI condition for the feasibility of the power flow equations under small perturbations of the power demands (Theorem 5.1). In addition we derive **(b)** the alternatives of these LMI conditions (Theorem 5.2).

M6. We give an alternative parametrization of \mathcal{D} , its closure and its boundary (Theorem 6.6), which is in a sense dual to the parametrization mentioned in **M1**.

M7. We give two parametrizations of $\partial\mathcal{F}$, the boundary of the set of feasible power demands (Theorem 6.8, Corollary 6.9).

M8. We consider the restriction of \mathcal{F} to nonnegative power demands, and present a parametrization for $\partial\mathcal{F}$ for nonnegative power demands **(a)** which is cheaper to compute than **M7**. Moreover, we deduce a refinement of the necessary and sufficient condition **M5** for nonnegative power demands **(b)** which provides an alternative to **M5** for determining power flow feasibility (Theorem 6.12, Theorem 6.15).

M9. We prove that any vector of power demands that is element-wise dominated by a feasible vector of power demands is also feasible (Lemma 7.1).

M10. We present two novel sufficient conditions for the feasibility of the power flow equations which generalize the sufficient conditions in [3] and [6] (Corollary 7.3, Theorem 7.5), and show how these conditions are related (Lemma 7.9).

M11. We show that the long-term voltage stable operating point is a strict high-voltage solution (Theorem 8.1). Consequently, the operating points associated to a feasible power demand which are either long-term

voltage stable, a high-voltage solution, or dissipation-minimizing, are one and the same (Theorem 8.2).

It is important to explain how these results are related to the existing literature, and in which regard these results are, to the best of the authors' knowledge, novel.

Regarding **M3**, it was shown in [4] that if the power flow equations are feasible, then there "almost surely" exists an operating point which is long-term voltage stable, and that it is the unique long-term voltage stable operating point if all power demands are positive, or all are negative. By studying long-term voltage *semi*-stable operating points, we show that for each feasible vector of power demands there always exists a unique long-term semi-stable operating point.

Regarding **M4**, the convexity of \mathcal{F} was already shown in [8] (see also [10]), and follows from an analysis of the convexity of the joint numerical range of a generalized quadratic form. Our approach to prove convexity is different from and less general than the one proposed by [8], and is a byproduct of the proof of **M3**, the one-to-one correspondence between \mathcal{F} and $\text{cl}(\mathcal{D})$. We believe our proof for convexity to be simpler. The exact description of all supporting half-spaces is novel.

Regarding **M5**, our contribution is a necessary and sufficient condition for the feasibility of power demands under small perturbations. In [7] a similar condition was shown to be sufficient for a more general system with constant-power loads at steady-state. It was shown in [7] to also be necessary whenever \mathcal{F} is closed convex, as is the case here.

Regarding **M11**, it was shown in [4] that, if the power flow equations are feasible, then there exists a high-voltage solution, *i.e.*, an operating point that element-wise dominates all other operating points which satisfy the power flow equations, and that this operating point is "almost surely" long-term voltage stable. We show that the element-wise domination is strict, and that this operating point always coincides with the unique long-term voltage semi-stable operating point. This shows the algorithm proposed in [4] converges to the unique long-term voltage semi-stable operating point wherever the power flow equations are feasible.

Organization of Part I

In Section II we formulate the DC power flow equations, discuss the problem of their feasibility, and define different types of desirable operating points. We give a detailed introduction to this feasibility problem and its difficulties, and motivate the problem by a dynamical DC power grid model.

In Section III we develop a geometric framework to analyze the DC power flow equations. The main objective of Section III is to prove that there is a one-to-one correspondence between \mathcal{F} and \mathcal{D} , and to give a method to compute the desired operating point (**M3**). In addition, we prove that \mathcal{F} is convex and present a full geometric characterization of \mathcal{F} as an intersection of half-spaces (**M4**). To establish this, we present a parametrization of \mathcal{D} (**M1**), and prove that there is a one-to-one correspondence between the boundary of \mathcal{D} and the boundary of the convex hull of \mathcal{F} , which is a prelude to

proving that there is a one-to-one correspondence between the boundary of \mathcal{D} and the boundary of \mathcal{F} (**M2**).

Section IV concludes Part I of the paper.

Notation and matrix definitions

For a vector $x = (x_1 \ \cdots \ x_k)^\top$ we denote

$$[x] := \text{diag}(x_1, \dots, x_k).$$

We let $\mathbb{1}$ and $\mathbb{0}$ denote the all-ones and all-zeros vector, respectively, and let I denote the identity matrix. We let their dimensions follow from their context. All vector and matrix inequalities are taken to be element-wise. We write $x \preceq y$ if $x \leq y$ and $x \neq y$. We let $\|x\|_p$ denote the p -norm of $x \in \mathbb{R}^k$.

We define $\mathbf{n} := \{1, \dots, n\}$. All matrices are square $n \times n$ matrices, unless stated otherwise. The submatrix of a matrix A with rows and columns indexed by $\alpha, \beta \subseteq \mathbf{n}$, respectively, is denoted by $A_{[\alpha, \beta]}$. The same notation $v_{[\alpha]}$ is used for subvectors of a vector v . We let α^c denote the set-theoretic complement of α with respect to \mathbf{n} . For a set S , the notation $\text{int}(S)$, $\text{cl}(S)$, ∂S and $\text{conv}(S)$ is used for the interior, closure, boundary and convex hull of S , respectively.

We list some classical definitions from matrix theory.

Definition 1.1 ([11], Ch. 5): A matrix A is a *Z-matrix* if $A_{ij} \leq 0$ for all $i \neq j$.

Definition 1.2 ([11], Thm. 5.3): A *Z-matrix* is an *M-matrix* if all its eigenvalues have nonnegative real part.

Definition 1.3 ([11], pp. 71): A matrix A is *irreducible* if for every nonempty set $\alpha \subsetneq \mathbf{n}$ we have $A_{[\alpha, \alpha^c]} \neq 0$.

II. THE DC POWER FLOW FEASIBILITY PROBLEM

In this section we formulate the DC power flow equations and explore their feasibility. A simple model of a dynamical DC power grid with constant-power loads is included to motivate the relevance of the current paper to the analysis and control of such power systems.

Our analysis applies to DC power grids that consist of nodes (buses), which are either loads or sources, and are interconnected by lines. Source nodes are voltage controlled buses which provide power to the power grid, and represent generators such as power plants. Load nodes are voltage controlled buses which generically extract power from the power grid. In this paper we do not consider operational limits on lines, currents, voltage potentials or power injections in the power grid. Including such operational limits further increases the complexity of the power flow feasibility problem.

We proceed with the modeling of DC power grids with purely resistive lines at steady state. We consider a DC power grid with n load nodes and m source nodes. We write $i \sim j$ if there exists a line between node i and node j , and $i \not\sim j$ otherwise. The conductance of the line between node i and node j is denoted by $w_{ij} = w_{ji}$, which is a positive real number. The Kirchhoff matrix $Y \in \mathbb{R}^{(n+m) \times (n+m)}$ describes both the topology and the conductances of the lines in the power grid. It is defined by

$$Y_{ij} := \begin{cases} \sum_{k \sim i} w_{ik} & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases}$$

Note that Y is a symmetric Z -matrix of which all rows and columns sum to zero. Hence $Y\mathbf{1} = \mathbf{0}$, and thus the matrix Y is singular. We assume that the nodes and lines form a connected graph. This implies that $\mathbf{1}$ spans the kernel of Y [12], and that all principal submatrices of Y are invertible [13]. One may verify that for vectors x, z we have

$$x^\top Yz = \sum_{i,j: i \sim j} w_{ij}(x_i - x_j)(z_i - z_j). \quad (1)$$

Since $w_{ij} > 0$ whenever $i \sim j$, (1) implies $x^\top Yx \geq 0$, and hence Y is positive semi-definite. Consequently, all principal submatrices of Y are positive definite.

We partition Y according to whether nodes are loads (L) or sources (S):

$$Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix}. \quad (2)$$

The matrices Y_{LL} and Y_{SS} are positive definite, as they are principal submatrices of Y . Following the same partition, let

$$V = \begin{pmatrix} V_L \\ V_S \end{pmatrix} \in \mathbb{R}^{n+m}$$

denote the vector of voltage potentials at the nodes. All voltage potentials are assumed to be positive (*i.e.*, $V > \mathbf{0}$).

We let $\mathcal{I} \in \mathbb{R}^{n+m}$ denote the electric current injected into the power grid at the nodes. The power that a node provides to the power grid is given by the vector

$$P = [V]\mathcal{I} \in \mathbb{R}^{n+m}.$$

Kirchhoff's laws together with Ohm's law state that

$$\begin{pmatrix} \mathcal{I}_L \\ \mathcal{I}_S \end{pmatrix} = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \begin{pmatrix} V_L \\ V_S \end{pmatrix}, \quad (3)$$

and therefore

$$\begin{pmatrix} P_L \\ P_S \end{pmatrix} = \begin{bmatrix} V_L \\ V_S \end{bmatrix} \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \begin{pmatrix} V_L \\ V_S \end{pmatrix}. \quad (4)$$

The *total dissipated power* in the lines is derived in [12] as

$$R(V_L, V_S) := V^\top YV \geq 0.$$

A. Feasible constant power demands

Throughout this paper we consider V_L as a variable of the system, whereas the Kirchhoff matrix Y and the voltages at the sources V_S are fixed. We therefore write $\mathcal{I}_L = \mathcal{I}_L(V_L)$ and $P_L = P_L(V_L)$.

Definition 2.1: We define the *source-injected currents* by

$$\mathcal{I}_L^* := -Y_{LS}V_S = -\mathcal{I}_L(\mathbf{0}), \quad (5)$$

which correspond to the currents injected into the loads by the sources when $V_L = \mathbf{0}$.

Proposition 2.2 ([12, Prop. 3.6]): The *open-circuit voltages* V_L^* are the unique voltage potentials at the loads so that $\mathcal{I}_L(V_L) = \mathbf{0}$, given by

$$V_L^* := -Y_{LL}^{-1}Y_{LS}V_S = Y_{LL}^{-1}\mathcal{I}_L^*. \quad (6)$$

Lemma 2.3: The source-injected currents are nonnegative and not all zero (*i.e.*, $\mathcal{I}_L^* \succeq \mathbf{0}$) and the open-circuit voltages V_L^* are positive (*i.e.*, $V_L^* > \mathbf{0}$).

Proof: Let $\alpha_1, \dots, \alpha_k \subseteq \mathbf{n}$ index the connected components of the graph formed by loads and the lines between them. The matrices $(Y_{LL})_{[\alpha_i, \alpha_i]}$ are therefore irreducible, while $(Y_{LL})_{[\alpha_i, \alpha_j]} = 0$ for $i \neq j$ [11]. Note that $(Y_{LL})_{[\alpha_i, \alpha_i]}$ is a principal submatrix of a Y , and is therefore positive definite, and that its off-diagonal elements are nonpositive. This implies that $(Y_{LL})_{[\alpha_i, \alpha_i]}$ is an irreducible nonsingular M -matrix, and therefore has a positive inverse [11, Thm. 5.12]. It follows that Y_{LL}^{-1} is (permutation similar to) a block diagonal matrix with positive diagonal blocks. The matrix $-Y_{LS}$ is a nonnegative matrix and the vector V_S is positive. Hence, $-Y_{LS}V_S = \mathcal{I}_L^*$ is nonnegative. Since the graph of Y is connected, there exists a line between the connected component represented by α_i and a source node. This implies that $(\mathcal{I}_L^*)_{[\alpha_i]}$ does not equal $\mathbf{0}$, and thus $(\mathcal{I}_L^*)_{[\alpha_i]} \succeq \mathbf{0}$. We conclude that the vector

$$(V_L^*)_{[\alpha_i]} = (Y_{LL}^{-1})_{[\alpha_i, \alpha_i]}(\mathcal{I}_L^*)_{[\alpha_i]} = (Y_{LL})_{[\alpha_i, \alpha_i]}^{-1}(\mathcal{I}_L^*)_{[\alpha_i]}$$

is a positive vector, since it is the product of a positive matrix and a nonnegative nonzero vector. This observation holds for all i , and thus $V_L^* > \mathbf{0}$ and $\mathcal{I}_L^* \succeq \mathbf{0}$. \square

If we substitute (6) into (4), we obtain the equation

$$P_L(V_L) = [V_L]Y_{LL}(V_L - V_L^*). \quad (7)$$

The open-circuit voltages are the unique voltage potentials at the loads which satisfy $V_L > \mathbf{0}$ and $P_L(V_L) = \mathbf{0}$.

In this paper we consider *constant-power loads*, which is to say that each load demands a fixed quantity of power from the power grid. We let $P_c \in \mathbb{R}^n$ denote the vector of constant power demands. Note that we do not impose any sign restrictions on P_c and that, in principle, load nodes could also demand negative power, in which case the loads provide constant power to the grid. Such cases are becoming more relevant due to the advent of prosumers in modern power grids. By equating the power demand with the power injection at the load nodes, we obtain the power balance

$$P_L(V_L) + P_c = \mathbf{0}. \quad (8)$$

Note that power demand and power injection have opposite signs, and that indeed the vectors in (8) should be summed. The substitution of (7) in (8) yields the *DC power flow equation for constant-power loads*:¹

$$[V_L]Y_{LL}(V_L - V_L^*) + P_c = \mathbf{0}. \quad (9)$$

Definition 2.4: Given Y and V_S , we say that the power flow equations (9) are *feasible* for a vector of power demands P_c if there exists a vector of voltage potentials V_L which satisfies (9). We say that V_L is an *operating point* associated to P_c if V_L satisfies (9) for P_c .

Recall that throughout Definition 2.4 we require that $V_S > \mathbf{0}$ and $V_L > \mathbf{0}$.

Related to the power flow feasibility problem is the power flow problem, which asks to find V_L and P_S such that (4) holds

¹Note that the terminology ‘‘DC power flow equation for constant-power loads’’ is also used in the literature to mean the linear approximation of the active power flow in an AC power grid by a DC current flow model [14], and does not coincide with (9).

with Y , V_S and P_L given. We remark that the feasibility of the power flow problem coincides with Problem 2.6. The power injections P_S are uniquely determined by the operating point V_L due to (4).

Definition 2.5: We say that a vector of power demands P_c is *feasible* if (9) is feasible for P_c . The set of *feasible power demands* is given by

$$\mathcal{F} := \{ P_c \mid \exists V_L > 0 \text{ s.t. } [V_L]Y_{LL}(V_L - V_L^*) + P_c = 0 \}.$$

Since (9) is a quadratic equation in V_L , the existence of a solution to (9) for a given P_c is not guaranteed. Furthermore, (9) may have multiple solutions, and multiple operating points for a single P_c may exist.

Our goal is to characterize all constant power demands P_c such that (9) is feasible, which is precisely the set \mathcal{F} . This is formalized in the following problem statement.

Problem 2.6: Consider a DC power grid with Kirchhoff matrix Y and voltage potentials at the sources $V_S > 0$. Let V_L^* be given by (6). For which power demands at the loads $P_c \in \mathbb{R}^n$ does there exist an operating point $V_L > 0$ of voltage potentials at the loads which satisfies (9)?

We make a few observations. First, note that Problem 2.6 is not affected by the lines between sources, since the matrix Y_{SS} does not appear in (9). More specifically, since (9) only depends on Y_{LL} , V_L^* , P_L and V_L , it follows that Problem 2.6 only depends on Y_{LL} and V_L^* , or on Y_{LL} and \mathcal{I}_L^* by (6). Second, if the graph formed by loads and the lines between them is not connected, then the matrix Y_{LL} is permutation similar to a block diagonal matrix with multiple blocks, as was observed in the proof of Lemma 2.3. It follows that (9) can be analyzed for each block separately. Hence, without loss of generality, we make the following assumption:

Assumption 2.7: The load nodes and the lines between loads form a connected graph, or equivalently by [11, Thm. 3.6.a], the matrix Y_{LL} is irreducible.

B. Desirable operating points

For a feasible power demand P_c there may be multiple operating points which satisfy (9). We are generically interested in the following two criteria to determine a desirable operating point.

a) Long-term voltage stable operating points: First, we desire that the selected operating point is such that a small increase in a single power demand leads to a small decrease in all voltage potentials [1], [3]. Note that each vector V_L of voltage potentials at the loads is associated by (9) to a vector of constant power demands P_c , given by

$$P_c(V_L) = [V_L]Y_{LL}(V_L^* - V_L). \quad (10)$$

We remark that (10) should be interpreted as the vector of constant power demand which are satisfied by the vector V_L of voltage potentials at steady state, and that constant power demands do not depend on the voltage potentials at the loads. By virtue of (10) the Jacobian of P_c at an operating point \tilde{V}_L is well-defined, and is given by

$$\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) = [Y_{LL}(V_L^* - \tilde{V}_L)] - [\tilde{V}_L]Y_{LL}. \quad (11)$$

We use the following definition from [3, Note 4, Def. 1].²

Definition 2.8: An operating point \tilde{V}_L associated to \tilde{P}_c is *long-term voltage stable* if the Jacobian of $P_c(V_L)$ at \tilde{V}_L is nonsingular, and its inverse is a matrix with negative elements (i.e.,³ $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} = \frac{\partial V_L}{\partial P_c}(\tilde{P}_c) < 0$).

The long-term voltage stability of an operating point describes the local property that an increase of a single constant-power load leads to a decrease of voltage potentials decrease at all loads. We refer to [3], [1], [16] for more complete interpretation of the long-term voltage stability of the operating point in power grids.

Long-term voltage stability may also be defined using Hurwitz stability of the Jacobian of P_c , due to [3, Note 4, Lem. 2]:

Proposition 2.9: The following are equivalent:

- An operating point \tilde{V}_L associated to \tilde{P}_c is long-term voltage stable;
- The Jacobian $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz stable.

We remark that multiple equivalent definitions for long-term voltage stability of the operating point can be found in the literature. For example, [3] refers to operating points described by Definition 2.8 as locally voltage stable, whereas [4] uses the term voltage-regularity. We refer to Remark 3.5 for a more detailed discussion.

The current paper studies the set of all long-term voltage stable operating points, which, by Proposition 2.9, equals

$$\mathcal{D} := \left\{ \tilde{V}_L \mid \frac{\partial P_c}{\partial V_L}(\tilde{V}_L) \text{ is Hurwitz stable} \right\}.$$

Similar to Definition 2.8 we define the notion of long-term voltage semi-stability:

Definition 2.10: An operating point \tilde{V}_L associated to \tilde{P}_c is *long-term voltage semi-stable* if for every $\varepsilon > 0$ there exists a long-term voltage stable operating point \hat{V}_L associated to some \hat{P}_c such that $\|\tilde{V}_L - \hat{V}_L\|_2 < \varepsilon$. Consequently, the set of all long-term voltage semi-stable operating points equals $\text{cl}(\mathcal{D})$, the closure of \mathcal{D} .

We emphasize that it is *a priori* not clear that each feasible P_c has a long-term voltage semi-stable operating point.

b) Dissipation-minimizing operating points: Second, it is desirable that an operating point V_L associated to P_c minimizes $R(V_L, V_S)$, the total power dissipated in the lines.

Definition 2.11: Given \tilde{P}_c , an operating point \tilde{V}_L associated to \tilde{P}_c is *dissipation-minimizing* if for all operating points V_L associated to \tilde{P}_c we have $R(\tilde{V}_L, V_S) \leq R(V_L, V_S)$.

For such operating points the following proposition applies.

Proposition 2.12: Given $\tilde{P}_c \in \mathcal{F}$, an operating point \tilde{V}_L associated to \tilde{P}_c is *dissipation-minimizing* if and only if it maximizes $V_L^\top \mathcal{I}_L^*$ among all operating points V_L associated to \tilde{P}_c , where \mathcal{I}_L^* is the quantity defined in (5).

²In [3] this property is referred to as local voltage stability, whereas we prefer the term long-term voltage stability.

³The equality $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} = \frac{\partial V_L}{\partial P_c}(\tilde{P}_c)$ holds locally and follows from the Inverse Function Theorem, see e.g. [15].

Proof: We define \mathcal{S} to be the set of all $V_L > 0$ which satisfies (9). We write $R(V_L, V_S) = V^\top YV$ in terms of the partitioning in (2) and substitute (6) and (5). This results in

$$\begin{aligned} R(V_L, V_S) &= V_L^\top (Y_{LL}V_L + Y_{LS}V_S) + V_S^\top (Y_{SS}V_S + Y_{SL}V_L) \\ &= V_L^\top Y_{LL}(V_L - V_L^*) + V_S^\top Y_{SS}V_S - V_L^\top \mathcal{I}_L^*. \end{aligned} \quad (12)$$

For any $V_L \in \mathcal{S}$, multiplying (9) by $\mathbb{1}^\top$ shows that

$$V_L^\top Y_{LL}(V_L - V_L^*) = -\mathbb{1}^\top P_c. \quad (13)$$

We are interested in minimizing $R(V_L, V_S)$ over all $V_L \in \mathcal{S}$. By using (12) and substituting (13), for all $V_L \in \mathcal{S}$ we have

$$R(V_L, V_S) = \underbrace{-\mathbb{1}^\top P_c + V_S^\top Y_{SS}V_S}_{\text{fixed}} - V_L^\top \mathcal{I}_L^*. \quad (14)$$

Only the last term in (14) depends on V_L , whereas the other terms are fixed. Thus, minimizing $R(V_L, V_S)$ over all $V_L \in \mathcal{S}$ is equivalent to maximizing $V_L^\top \mathcal{I}_L^*$ over all $V_L \in \mathcal{S}$. \square

It was shown in [4] that for each feasible vector of power demands there exists an operating point associated to a given P_c which element-wise dominates all other operating points associated to P_c . Such an operating point is referred to as the *high-voltage solution* to (9) (see also [3]). We formalize this notion by the following definition.

Definition 2.13: An operating point \tilde{V}_L associated to the power demands P_c is a *high-voltage solution* if $\tilde{V}_L \geq V_L$ for all V_L associated to P_c , and is a *strict high-voltage solution* if $\tilde{V}_L > V_L$ for all $V_L \neq \tilde{V}_L$ associated to P_c .

It follows from Proposition 2.12 that a high-voltage solution is always dissipation-minimizing.

Corollary 2.14: If the operating point \tilde{V}_L associated to the power demands P_c is a high-voltage solution, then \tilde{V}_L is dissipation-minimizing. Moreover, if \tilde{V}_L is a strict high-voltage solution, then \tilde{V}_L is the unique dissipation-minimizing operating point.

Proof: Let V_L be an operating point associated to P_c . If $\tilde{V}_L - V_L \geq 0$ then $(\tilde{V}_L - V_L)^\top \mathcal{I}_L^* \geq 0$ since $\mathcal{I}_L^* \geq 0$ by Lemma 2.12, and hence $\tilde{V}_L^\top \mathcal{I}_L^* \geq V_L^\top \mathcal{I}_L^*$. If $\tilde{V}_L - V_L > 0$ then $(\tilde{V}_L - V_L)^\top \mathcal{I}_L^* > 0$ since $\mathcal{I}_L^* \geq 0$ and $\mathcal{I}_L^* \neq 0$, and hence $\tilde{V}_L^\top \mathcal{I}_L^* > V_L^\top \mathcal{I}_L^*$. The result follows from Proposition 2.12. \square

Note that Definitions 2.8 and 2.10 describe local properties of an operating point, while Definitions 2.11 and 2.13 are global properties concerning all operating points associated to P_c . It is *a priori* not clear how dissipation-minimizing operating points and long-term voltage semi-stable operating points are related, nor is it clear when a feasible vector of power demands has a (possibly unique) long-term voltage semi-stable operating point or when it has a (strict) high-voltage solution. Some partial answers to these questions are known—see [4], [3]—but a full characterization is lacking. This fundamental question is answered in this paper. Indeed, in Part I of this paper we show that for each feasible vector of power demands there exists a unique long-term voltage semi-stable operating point associated to P_c (**M3**). In Part II we show that this operating point is a strict high-voltage solution (**M11**), which proves that the aforementioned notions coincide due to Corollary 2.14.

C. A motivating example of a dynamical power grid

To illustrate how the present paper is relevant in a dynamical setting we consider a simple model of a dynamical DC power grid with RL-lines and capacitors and constant-power components at the loads. We refer to [4], [3] for other examples of power grids for which our analysis applies. We let l represent the number of transmission lines in the power grid. We number the lines and write $k \leftrightarrow \{i, j\}$ if the k -th edge corresponds to the line between nodes i and j . We consider a power grid with RL transmission lines. The conductance and inductance of line k are denoted by G_k and L_k , respectively, which are positive real numbers. This means that if $k \leftrightarrow \{i, j\}$ we have $G_k = G_{ij} = G_{ji}$. We have $G, L \in \mathbb{R}^l$. We define the node-edge incidence matrix $B \in \{1, -1, 0\}^{(n+m) \times l}$ by

$$B_{ik} = \begin{cases} 1 & \text{if } k \leftrightarrow \{i, j\} \text{ for some } j, \text{ and } i < j \\ -1 & \text{if } k \leftrightarrow \{i, j\} \text{ for some } j, \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

We partition the rows of B according to whether nodes are loads or sources by $B = (B_L^\top \ B_S^\top)^\top$. At each load we attach a shunt capacitor and a constant-power load component in parallel. The shunt capacitances are positive and are collected in the vector $C \in \mathbb{R}^n$, and the constant-power demands are represented by $P_c \in \mathbb{R}^n$ as before. The voltage potentials over these components are given by V_L . Circuit theory tells us that the dynamics of this system are given by

$$\begin{cases} [L]\dot{\mathcal{I}}^{\text{lines}} &= -[G]^{-1}\mathcal{I}^{\text{lines}} + B_L^\top V_L + B_S^\top V_S \\ [C]\dot{V}_L &= B_L\mathcal{I}^{\text{lines}} + [V_L]^{-1}P_c \end{cases}, \quad (15)$$

where $\mathcal{I}^{\text{lines}} \in \mathbb{R}^l$ represent the current flow through the lines and the term $-[V_L]^{-1}P_c$ corresponds to the constant-power component. Note that V_S acts as an input to the system.

Suppose we want to design an input signal V_S to steer the state to an equilibrium. Since (15) is a nonlinear system due to the term $-[V_L]^{-1}P_c$, the analysis of the existence and stability of such equilibria is nontrivial and is crucial for designing V_S . Assuming that the power grid is at steady state, we have that the left-hand side of (15) vanishes, and after eliminating $\mathcal{I}^{\text{lines}}$ we are left with the algebraic equation

$$B_L[G]B_L^\top V_L + B_L[G]B_S^\top V_S + [V_L]^{-1}P_c = 0, \quad (16)$$

which should hold at steady state. It can be shown that $Y = B[G]B^\top$ holds, and by substituting this relation into (16), and substituting (6) and multiplying by $[V_L]$, it is observed that the solvability of (16) and (9) are equivalent. Hence the existence of equilibria to (15) is equivalent to Problem 2.6, and is necessary for the design of the input V_S .

To assess the stability at an equilibrium of (15) we study the stability of the Jacobian associated to (15), given by

$$\begin{pmatrix} [L]^{-1} & 0 \\ 0 & [C]^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} -[G]^{-1} & B_L^\top \\ B_L & [V_L]^{-2}[P_c] \end{pmatrix}}_{=:U}. \quad (17)$$

Since U is symmetric and $[L]^{-1}$ and $[C]^{-1}$ are positive definite, (17) is Hurwitz stable if and only if U is negative definite. Since $-[G]^{-1}$ is negative definite, it follows by the

Haynsworth inertia additivity formula ([17], Sec. 0.10) that U is negative definite if and only if the Schur complement of U

$$\begin{aligned} & [V_L]^{-2}[P_c] - B_L[G]B_L^\top \\ & = [V_L]^{-1}[Y_{LL}(V_L^* - V_L)] - Y_{LL} = [V_L]^{-1}\frac{\partial P_c}{\partial V_L}(V_L) \end{aligned} \quad (18)$$

is negative definite, where we used $Y = B[G]B^\top$ and substituted (9) and (11). If (18) is negative definite, then $\frac{\partial P_c}{\partial V_L}(V_L)$ is Hurwitz since $[V_L]^{-1}$ is positive definite. The above derivation shows that an equilibrium $(\mathcal{I}^{\text{lines}}, V_L)$ of (15) is Hurwitz stable if and only if the corresponding operating point V_L is long-term voltage stable. A similar argument can be made for Hurwitz semi-stability and long-term voltage semi-stability.

In conclusion, the above discussion presents an example of a dynamical DC power grid model for which the existence of an equilibrium corresponds to power flow feasibility, and for which long-term voltage stability of the operating point captures a dynamic stability of the equilibria. Similar observations can be made for other power systems; we refer to [4], [3].

D. Academic examples of power flow feasibility

In this section we explore the intricacies of Problem 2.6 by considering two simple examples. We will focus on building some intuition for the sets \mathcal{F} and \mathcal{D} . We first consider the simplest case of a DC power grid with constant-power loads.

Example 2.15 (Single load case): Consider a DC power grid with a single load and a single source (i.e., $n = m = 1$), as depicted in Figure 1. The corresponding graph of $P_L(V_L)$ is given in Figure 2. Figure 3 depicts the relation between P_c and V_L . In this example we let $V_S > 0$ be arbitrary and recall that the open-circuit voltages $V_L^* > 0$ are defined by (6). Since $n = 1$, it follows that (9) is scalar-valued. By taking (9) and completing the squares we find

$$Y_{LL}(V_L - \frac{1}{2}V_L^*)^2 - \frac{1}{4}Y_{LL}(V_L^*)^2 + P_c = 0.$$

Since $n = 1$, Y_{LL} is a positive scalar, and it follows that

$$V_L = \frac{1}{2}V_L^* \pm \sqrt{Y_{LL}^{-1}(\frac{1}{4}Y_{LL}(V_L^*)^2 - P_c)}. \quad (19)$$

We see that (9) has a real solution for $n = 1$ if and only if

$$P_c \leq \frac{1}{4}Y_{LL}(V_L^*)^2. \quad (20)$$

The set of all feasible power demands is therefore given by

$$\mathcal{F} = \{ P_c \mid P_c \leq \frac{1}{4}Y_{LL}(V_L^*)^2 \}. \quad (21)$$

If equality holds in (20), it follows from (19) that there is precisely one operating point, given by $V_L = \frac{1}{2}V_L^*$. In the case that (20) is strict, we see that the positive branch of (19) leads to a higher voltage potential at the load, and minimizes $R(V_L, V_S)$. Hence, the positive branch of (19) is the high-voltage solution. In addition, the positive branch decreases when P_c increases, and so it is also the long-term voltage stable solution. We have $\mathcal{D} = \{ V_L \mid V_L > \frac{1}{2}V_L^* \}$. \square

Eq. (20) of Example 2.15 shows that (9) is not always feasible for each P_c for $n = 1$. We will show that the same

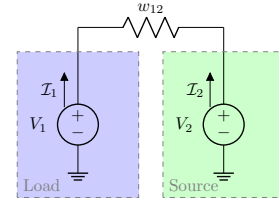


Fig. 1. A schematic depiction of a power grid with a single load node and a single source node ($n = m = 1$), where w_{12} is the conductance of the line between the nodes.

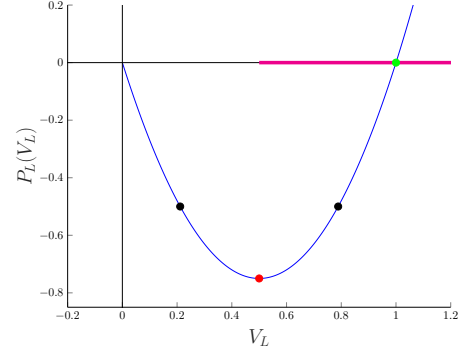


Fig. 2. A plot of $P_L(V_L)$ against V_L for the power grid in Figure 1 with $w_{12} = 3 \Omega^{-1}$ and $V_S = V_2 = 1$ V. The red point indicates the voltage such that the power that the grid transports is maximized. The thick purple half-line corresponds to \mathcal{D} , the set of all long-term voltage stable operating points. The black dots are voltage potentials which result in the same power transfer to the load. The green dot corresponds to the open-circuit voltages, for which we observe that the power transfer is zero.

is true for $n > 1$ by studying the maximal total amount of power that can be transported to the load nodes.

Definition 2.16: For a feasible power demand $P_c \in \mathcal{F}$, the *total feasible power demand* is the sum $\mathbb{1}^\top P_c$ of the power demands at the loads.

Definition 2.17: A *maximizing feasible power demand* is a feasible power demand $P_{\max} \in \mathcal{F}$ that maximizes the total

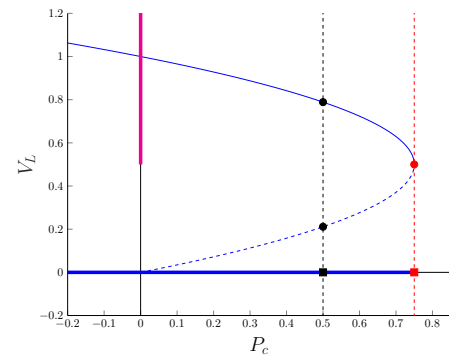


Fig. 3. A plot of V_L against P_c for the power grid in Figure 1 with $w_{12} = 3 \Omega^{-1}$ and $V_S = V_2 = 1$ V. The thick blue half-line corresponds to all feasible power demands at the load node. The red square indicates the maximal power that the load can drain from the power grid. There are multiple operating points if $0 < P_c < 0.75$, as is the case for the black points. The Jacobian $\frac{\partial V_L}{\partial P_c}$ is not defined in the red point. The corresponding operating point is long-term voltage semi-stable, but not long-term voltage stable. The solid curve depicts a one-to-one correspondence between the feasible power demands (the thick blue half-line) and the long-term voltage semi-stable operating points (the thick purple half-line).

feasible power demand. Thus for all $P_c \in \mathcal{F}$ it satisfies

$$\mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max}. \quad (22)$$

Lemma 2.18: There is a unique maximizing feasible power demand $P_{\max} \in \mathcal{F}$. It is given by

$$P_{\max} = \frac{1}{4}[V_L^*]^\top \mathcal{I}_L^* \geq 0. \quad (23)$$

The unique operating point corresponding to P_{\max} is $\frac{1}{2}V_L^*$.

Proof: Let P_c be feasible, and let V_L be an associated operating point. Recall from (13) that the total feasible power demand $\mathbb{1}^\top P_c$ satisfies

$$\mathbb{1}^\top P_c = -V_L^\top Y_{LL}(V_L - V_L^*).$$

By completing the squares we find that

$$\mathbb{1}^\top P_c = -(V_L - \frac{1}{2}V_L^*)^\top Y_{LL}(V_L - \frac{1}{2}V_L^*) + \frac{1}{4}V_L^{*\top} Y_{LL} V_L^*.$$

Since Y_{LL} is positive definite, it follows that

$$\mathbb{1}^\top P_c \leq \frac{1}{4}V_L^{*\top} Y_{LL} V_L^*, \quad (24)$$

with equality if and only if $V_L = \frac{1}{2}V_L^*$. This implies that equality in (24) holds if and only if

$$P_c = -P_L(\frac{1}{2}V_L^*) = \frac{1}{4}[V_L^*]^\top Y_{LL} V_L^* = \frac{1}{4}[V_L^*]^\top \mathcal{I}_L^*,$$

where we have substituted (6). The above implies that there is a unique P_{\max} given by (23), and corresponds to the unique operating point $\frac{1}{2}V_L^*$. Lemma 2.3 implies that $P_{\max} \geq 0$, since $V_L^* > 0$ and $\mathcal{I}_L^* \geq 0$. \square

We remark that if a load node i does not share a line with a source node, then $(\mathcal{I}_L^*)_i = (-Y_{LS}V_S)_i = 0$, and $P_{\max,i} = 0$.

The inequality (22) describes a closed half-space in the space of power demands, and is a necessary condition for the feasibility of (9). This condition coincides with the inclusion

$$\mathcal{F} \subseteq \{ P_c \mid \mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max} \}. \quad (25)$$

We observe that (25) generalizes (21) for $n \geq 1$. Since there is a unique maximizing feasible power demand by Lemma 2.18, equality in (22) only holds for P_{\max} , and the inclusion in (25) strict for $n > 1$.

The converse of (22) states that, if P_c is such that $\mathbb{1}^\top P_c > \mathbb{1}^\top P_{\max}$, then no solution to (9) exists. The existence of P_{\max} therefore once more shows that the DC power flow equations with constant-power loads are not always feasible. The following example illustrates (25).

Example 2.19 (two loads, one source case): Consider the DC power grid with two loads ($n = 2$) and a source ($m = 1$); see Figure 4. Figure 5 gives the feasible power demands when

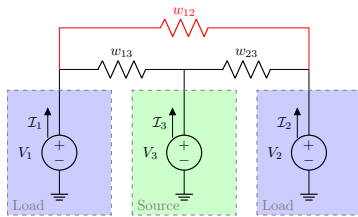


Fig. 4. A schematic depiction of a power grid with two load nodes ($n = 2$, $m = 1$).

we let $w_{13} = 3\Omega^{-1}$, $w_{23} = 2\Omega^{-1}$ and $V_S = 1\text{ V}$, and vary the conductance w_{12} . It can be shown that $V_L^* = \mathbb{1}V$.

First, we disregard the red line between node 1 and node 2 (*i.e.*, $1 \not\sim 2$), or equivalently take $w_{12} = 0\Omega^{-1}$. The absence of the red line implies that Y_{LL} is (block) diagonal, and Problem 2.6 reduces to two copies of Example 2.15. From (20) it follows that P_c is feasible if and only if $P_{c,1} \leq 0.75$ and $P_{c,2} \leq 0.5$, which corresponds to the blue rectangle in Figure 5. Next, we consider the red line between loads 1 and 2. We observe from the same figure that increasing w_{12} will result in a larger set of feasible power demands, as indicated by the green and yellow areas. The dashed line are the points for which equality in (22) holds. We note that these sets lie below the dashed line, and intersect the line only at P_{\max} , which illustrates (25). \square

Figure 6 relates the sufficient conditions of [3] and [6] to the feasible power demands of the DC power grid depicted in Figure 4. Figures 5 and 6 suggest some properties of the set of feasible power demands \mathcal{F} , which we will prove in this paper:

- The set \mathcal{F} is convex (**M4**) (See also [8]).
- Each hyperplane which is tangent to the boundary of \mathcal{F} (such as the dashed line in Figure 5) gives a necessary condition for feasibility (**M4**) (See also [7]).
- If $y \in \mathcal{F}$ and $\hat{y} \leq y$, then also $\hat{y} \in \mathcal{F}$ (**M9**).
- The convex hull of the points on the boundary of \mathcal{F} lead to a sufficient condition for a power demand to be feasible. In particular, the convex hull of 0 , P_{\max} and the points where the axes intersect the boundary of \mathcal{F} forms a polyhedral subset of \mathcal{F} . The interior of this set describes the sufficient condition from [3] (**M10**). See also Figure 6.
- The ball with the smallest radius such that it touches the boundary of the condition in [3] is contained in \mathcal{F} . This subset describes the sufficient condition of [7] (**M10**); see again Figure 6.

We also observe in Figure 5 that P_{\max} does not change when w_{12} is changed, and increasing w_{12} leads to nested⁴ sets \mathcal{F} of feasible power demands. We remark that this is not true in

⁴Nested with respect to inclusion.

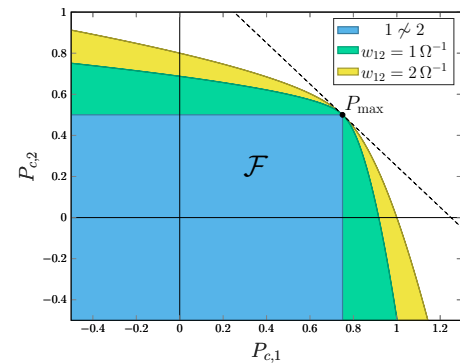


Fig. 5. Plots of \mathcal{F} , the set of feasible power demands, for the power grid in Figure 4 with different values of w_{12} , where $w_{13} = 3\Omega^{-1}$, $w_{23} = 2\Omega^{-1}$ and $V_S = 1\text{ V}$. The dashed line is the set of the points for which equality in (22) holds.

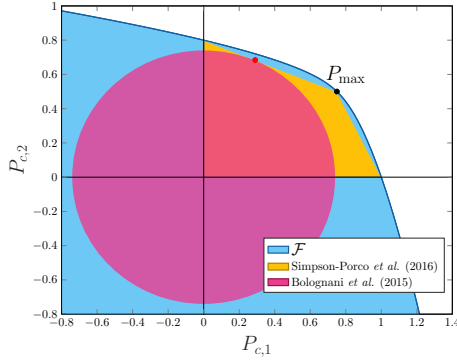


Fig. 6. A plot of the set \mathcal{F} for the power grid in Figure 4, where $w_{12} = 2 \Omega^{-1}$, $w_{13} = 3 \Omega^{-1}$, $w_{23} = 2 \Omega^{-1}$ and $V_S = 1 \text{ V}$. The yellow area is the set described by the sufficient condition in [3]. The red area is the sufficient condition from [6]. The boundaries of the two conditions intersect in the red point.

general. The analysis of this phenomenon is beyond the scope of this paper.

III. A GEOMETRIC FRAMEWORK FOR DC POWER FLOW FEASIBILITY WITH CONSTANT-POWER LOADS

In this section we establish a geometric framework for the feasibility of DC power flow with constant-power loads. This section is structured as follows. In Section III-A we show that every operating point is uniquely associated to a Z-matrix: the Jacobian of P_c at that operating point. Moreover, we show that an operating point is long-term voltage stable if and only if the Jacobian of P_c is a nonsingular M-matrix. Section III-B uses this characterization to obtain a parametrization of the set of long-term voltage stable operating points \mathcal{D} . In particular, we parametrize the boundary of \mathcal{D} by a set Λ_1 . In Section III-C we study the convex hull of \mathcal{F} and show that Λ_1 also parametrizes the boundary of $\text{conv}(\mathcal{F})$. This establishes a one-to-one correspondence between the boundary of \mathcal{D} and the boundary of $\text{conv}(\mathcal{F})$. Our main results are stated in Section III-D, in which we prove that for each feasible power demand there exists a unique long-term voltage semi-stable operating point. In addition we present an explicit method for computing this operating point and show that the set of feasible power demands is closed and convex.

A. Relating operating points to the Jacobian of P_c

Recall that Z-matrices, M-matrices and irreducible matrices were defined in Definitions 1.1-1.3.

Proposition 3.1: Let A be an irreducible Z-matrix. There is a unique eigenvalue r of A with smallest (*i.e.*, “most negative”) real part. The eigenvalue r , known as the *Perron root*, is real and simple. A corresponding eigenvector v , known as a *Perron vector*, is unique up to scaling, and can be chosen so that $v > 0$.

We include a short proof, as we were unable to find a reference to this result in this exact formulation.

Proof: Let B be a nonnegative matrix and s a scalar such that $A = sI - B$. Irreducibility is independent of the diagonal elements of a matrix. Hence, since A is irreducible, so is B . Let $\rho(B)$ denote the spectral radius of B . By the

Perron-Frobenius Theorem [11, Thm. 4.8], $\rho(B)$ is a simple eigenvalue of B and there exists a positive eigenvector v such that $Bv = \rho(B)v$. Hence, $r = s - \rho(B)$ is a simple eigenvalue of A . The corresponding eigenvector v is unique up to scaling. \square

Appendix A lists a number of useful results concerning Z-matrices, M-matrices and irreducible matrices.

Recall that the Jacobian of P_c at \tilde{V}_L is given by (11), and that a vector \tilde{V}_L qualifies as a vector of voltage potentials only if $\tilde{V}_L > 0$. The following lemma shows that the matrix (11) has a particular structure if (and only if) $\tilde{V}_L > 0$, and that each such matrix is unique for $\tilde{V}_L > 0$.

Lemma 3.2: The matrix $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ (*i.e.*, the Jacobian of $-P_c$ at \tilde{V}_L) is an irreducible Z-matrix if and only if $\tilde{V}_L > 0$. The map $\tilde{V}_L \mapsto \frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is injective for $\tilde{V}_L > 0$.

Proof: (\Rightarrow): Let $i \in \mathbf{n}$ and note that we have

$$\begin{aligned} \left(-\frac{\partial P_c}{\partial \tilde{V}_L}(x)\right)_{[i, i^c]} &= ([x]Y_{LL} - [Y_{LL}(V_L^* - x)])_{[i, i^c]} \\ &= ([x]Y_{LL})_{[i, i^c]} = x_i(Y_{LL})_{[i, i^c]}. \end{aligned} \quad (26)$$

If $x_i = 0$, then $\left(-\frac{\partial P_c}{\partial \tilde{V}_L}(x)\right)_{[i, i^c]} = 0$ by (26), which violates the irreducibility of the $-\frac{\partial P_c}{\partial \tilde{V}_L}(x)$. Hence $x_i \neq 0$. Since Y_{LL} is a Z-matrix, we have $(Y_{LL})_{[i, i^c]} \leq 0$. Since Y_{LL} is irreducible we know that $(Y_{LL})_{[i, i^c]} \neq 0$, by definition. It follows that there exists at least one negative element in $(Y_{LL})_{[i, i^c]}$. Hence, there exists a $j \neq i$ so that $(Y_{LL})_{ij} < 0$. It follows from (26) that

$$\left(-\frac{\partial P_c}{\partial \tilde{V}_L}(x)\right)_{ij} = x_i(Y_{LL})_{ij}. \quad (27)$$

The right-hand side of (27) is nonzero, $(Y_{LL})_{ij}$ is negative, and the left-hand side of (27) is nonpositive since $-\frac{\partial P_c}{\partial \tilde{V}_L}(x)$ is a Z-matrix. This implies that x_i is positive. Hence $x > 0$.

(\Leftarrow): The matrix Y_{LL} is an irreducible Z-matrix. Since $x > 0$, also $[x]Y_{LL}$ is an irreducible Z-matrix, by 3 and 4 of Proposition A.3. Consequently, $[x]Y_{LL} - [Y_{LL}(V_L^* - x)]$ is an irreducible Z-matrix, by 1 and 2 of Proposition A.3.

Let $x, z > 0$ satisfy $\frac{\partial P_c}{\partial \tilde{V}_L}(x) = \frac{\partial P_c}{\partial \tilde{V}_L}(z)$. By (27) this implies that for all i there exists a j such that $x_i(Y_{LL})_{ij} = z_i(Y_{LL})_{ij}$. Since $(Y_{LL})_{ij} \neq 0$, we have $x_i = z_i$, and hence $x = z$. \square

Lemma 3.2 states that the Jacobian of P_c at an operating point $\tilde{V}_L > 0$ is unique to \tilde{V}_L . This implies that operating points are uniquely identified by properties of the associated Jacobian. The following result identifies all long-term voltage stable operating points (see Definition 2.8) by means of properties of the associated Jacobian.

Proposition 3.3: The set \mathcal{D} of long-term voltage stable operating points equals

$$\mathcal{D} = \left\{ \tilde{V}_L > 0 \mid -\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L) \text{ is a nonsingular M-matrix} \right\}.$$

Proof: Let $\tilde{V}_L > 0$ be an operating point associated to some vector of power demands \tilde{P}_c . Recall from Definition 2.8 that \tilde{V}_L is long-term voltage stable if $\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is nonsingular and $\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^{-1} = \frac{\partial \tilde{V}_L}{\partial \tilde{P}_c}(\tilde{P}_c)$ is a matrix with negative elements. Since $\tilde{V}_L > 0$, $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is a Z-matrix by Lemma 3.2.

If follows by [11, Thm. 5.12] that $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^{-1} > 0$ if and only if $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is a nonsingular M-matrix. \square

Recall from Definition 2.10 that an operating point is long-term voltage semi-stable if it lies in the the closure of \mathcal{D} . Proposition 3.3 implies the following characterization of such operating points.

Corollary 3.4: The closure and boundary of \mathcal{D} satisfies

$$\begin{aligned} \text{cl}(\mathcal{D}) &= \left\{ \tilde{V}_L > 0 \mid -\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L) \text{ is an M-matrix} \right\}; \\ \partial\mathcal{D} &= \left\{ \tilde{V}_L > 0 \mid -\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L) \text{ is a singular M-matrix} \right\}. \end{aligned}$$

The proof follows directly from Proposition A.2.

Remark 3.5: Many equivalent characterizations of long-term voltage stable operating points may be derived from Proposition 3.3. Indeed, the paper [18] lists numerous equivalent conditions for when a Z-matrix is a nonsingular M-matrix. In particular, it follows from property J₂₉ of [18] together with Proposition A.3.5 that if $\tilde{V}_L > 0$, and hence $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is a Z-matrix by Lemma 3.2, then $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is a nonsingular M-matrix if and only if $[\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is Hurwitz stable. This shows that $\tilde{V}_L > 0$ is long-term voltage stable if and only if $[\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is Hurwitz stable. The latter property coincides with the definition of voltage-regularity found in [4] and shows up in Section II-C. Alternatively, without invoking Proposition A.3.5 it follows that $\tilde{V}_L > 0$ is long-term voltage stable if and only if $\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is Hurwitz stable, by which one may prove Proposition 2.9. Similarly it can be shown that $\tilde{V}_L > 0$ is long-term voltage semi-stable if and only if $\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)$ is Hurwitz semi-stable⁵.

B. A parametrization of \mathcal{D}

Proposition 3.3 and Corollary 3.4 allow us to deduce a parametrization for the set \mathcal{D} of all long-term voltage stable operating points. Such a parametrization gives a constructive method to determine where such operating points lie in the voltage domain, as opposed to testing at which operating points of interest the Jacobian of P_c is Hurwitz stable (see Remark 3.5).

We introduce the following definitions. For a vector $\lambda \in \mathbb{R}^n$ we introduce the $n \times n$ matrix

$$h(\lambda) := \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda]).$$

Note that $h(\mathbf{1}) = Y_{LL}$, and that

$$x^\top h(\lambda)x = x^\top [\lambda]Y_{LL}x = \lambda^\top [x]Y_{LL}x. \quad (28)$$

In addition we define the set

$$\Lambda := \{ \lambda \mid h(\lambda) \text{ is positive definite} \}. \quad (29)$$

The set Λ is studied in Appendix B. In particular, Lemma B.5 shows that Λ is convex, and Lemma B.6 shows that Λ lies in the positive orthant.

⁵By a Hurwitz semi-stable matrix we mean a matrix for which all its eigenvalues have negative real part, with the possible exception of a semi-simple eigenvalue 0. The eigenvalue 0 of a singular symmetric M-matrices is semisimple.

The following results extends Lemma 3.2 and explicitly relates any operating point to the Perron root and Perron vector of a Jacobian matrix at that operating point. This allows us to obtain a parametrization of the sets \mathcal{D} , $\text{cl}(\mathcal{D})$, and $\partial\mathcal{D}$.

Lemma 3.6: Let $r \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$ such that $r \geq 0$ and $\lambda > 0$. The Jacobian $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$ is an irreducible M-matrix with Perron root r and Perron vector λ if and only if $h(\lambda)$ is positive definite (i.e., $\lambda \in \Lambda$) and \tilde{V}_L satisfies

$$\tilde{V}_L = \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbf{1}), \quad (30)$$

in which case we have $\tilde{V}_L > 0$.

Proof: (\Rightarrow): The matrix Y_{LL} is an irreducible Z-matrix and $\lambda > 0$, and so $h(\lambda)$ is an irreducible Z-matrix by Propositions A.3 and A.4. We let s and $v > 0$ denote respectively the Perron root and a Perron vector of $h(\lambda)$. The matrix $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$ is an M-matrix, and hence a Z-matrix. By Lemma 3.2 we have $\tilde{V}_L > 0$. Using the fact that (r, λ) is an eigenpair to $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$, we observe that

$$\begin{aligned} r\lambda &= -\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top \lambda = -([Y_{LL}(V_L^* - \tilde{V}_L)] - Y_{LL}[\tilde{V}_L])\lambda \\ &= -[\lambda]Y_{LL}V_L^* + [\lambda]Y_{LL}\tilde{V}_L + Y_{LL}[\lambda]\tilde{V}_L \\ &= -[\lambda]\mathcal{I}_L^* + 2h(\lambda)\tilde{V}_L. \end{aligned}$$

By rearranging the terms, it follows that

$$[\lambda]\mathcal{I}_L^* + r\lambda = 2h(\lambda)\tilde{V}_L. \quad (31)$$

Multiplying (31) by v^\top results in

$$v^\top([\lambda]\mathcal{I}_L^* + r\lambda) = 2v^\top h(\lambda)\tilde{V}_L = 2sv^\top \tilde{V}_L. \quad (32)$$

Since $\tilde{V}_L > 0$, $v > 0$, $\lambda > 0$, $r \geq 0$ and $\mathcal{I}_L^* \succeq 0$, it follows that the left hand side of (32) is positive. Since $v^\top \tilde{V}_L$ is also positive, it follows from (32) that the Perron root s is positive. Hence, $h(\lambda)$ is a nonsingular M-matrix by Proposition A.2, and (30) follows from (31). Since $h(\lambda)$ is a symmetric nonsingular M-matrix, it is positive definite and $\lambda \in \Lambda$.

(\Leftarrow): If $\lambda \in \Lambda$, then $\lambda > 0$ by Lemma B.6. The rest of the proof follows by reversing the steps of the " \Rightarrow "-part. \square

Lemma 3.6 allows for an explicit parametrization of the set \mathcal{D} by Λ , and without relying on properties of the Jacobian of P_c . Note that (30) is invariant under scaling of λ , and hence the vectors λ may be normalized. For this purpose we define

$$\Lambda_1 := \Lambda \cap \{ \lambda > 0 \mid \|\lambda\|_1 = \mathbf{1}^\top \lambda = 1 \}, \quad (33)$$

which is a convex set, as it is the intersection of convex sets. Appendix B lists several properties of the closure of Λ_1 .

Theorem 3.7 (MI): The set \mathcal{D} of all long-term voltage stable operating point, its closure $\text{cl}(\mathcal{D})$ and its boundary $\partial\mathcal{D}$ are parametrized by

$$\begin{aligned} \mathcal{D} &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbf{1}) \mid \lambda \in \Lambda_1, r > 0 \right\} \\ \text{cl}(\mathcal{D}) &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbf{1}) \mid \lambda \in \Lambda_1, r \geq 0 \right\} \\ \partial\mathcal{D} &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda]\mathcal{I}_L^* \mid \lambda \in \Lambda_1 \right\}. \end{aligned}$$

Furthermore, the map

$$(\lambda, r) \mapsto \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbf{1}) \quad (34)$$

from $\Lambda_1 \times \mathbb{R}_{\geq 0}$ to $\text{cl}(\mathcal{D})$ is a bicontinuous map, and the sets \mathcal{D} , $\text{cl}(\mathcal{D})$ and $\partial\mathcal{D}$ are simply connected.

Proof: Proposition A.2 states that a Z-matrix is a (non-singular/singular) M-matrix if and only if its Perron root r is nonnegative (positive/zero). Proposition 3.3 and Corollary 3.4 together with Lemma 3.6 imply that the vector (30) with $\lambda \in \Lambda$ lies in \mathcal{D} , $\text{cl}(\mathcal{D})$ or $\partial\mathcal{D}$ if and only if r in (30) satisfies respectively $r > 0$, $r \geq 0$ or $r = 0$.

The map (34) is a continuous bijection from $\Lambda_1 \times \mathbb{R}_{\geq 0}$ to $\text{cl}(\mathcal{D})$, which follows from Lemma 3.6 and Corollary 3.4. The inverse of the map (34) is described taking $x \in \text{cl}(\mathcal{D})$ and computing the Perron vector $\lambda > 0$ and the Perron root r of $-\frac{\partial P_c}{\partial V_L}(x)$. By Proposition A.1, the Perron root and Perron vector of $-\frac{\partial P_c}{\partial V_L}(x)$ are continuous in x . Hence the inverse of the map (34) is also continuous.

The set $\Lambda_1 \times \mathbb{R}_{\geq 0}$ is convex, and is therefore simply connected, which is a topological property. Topological properties are preserved by bicontinuous maps, and thus $\text{cl}(\mathcal{D})$ is also simply connected. The same holds for $\Lambda_1 \times \mathbb{R}_{> 0}$ and $\Lambda_1 \times \{0\}$, and hence \mathcal{D} and $\partial\mathcal{D}$ are simply connected. \square

C. The convex hull of \mathcal{F} and its boundary

In order to study the set \mathcal{F} of feasible power demands, we will be studying its convex hull and its boundary. In particular, we show that there is a one-to-one correspondence between points in the boundary of \mathcal{D} and points in $\partial\text{conv}(\mathcal{F})$, the boundary of the convex hull of \mathcal{F} .

To simplify notation, we define for $\lambda \in \Lambda$ the map

$$\varphi(\lambda) := \frac{1}{2}h(\lambda)^{-1}[\lambda]I_L^*. \quad (35)$$

Note that $\varphi(\lambda)$ is invariant under scaling of λ .

The boundary of \mathcal{D} is by definition the set of the operating points which are long-term voltage semi-stable, but not long-term voltage stable. It follows from Theorem 3.7 that $\partial\mathcal{D}$ satisfies

$$\partial\mathcal{D} = \{ \varphi(\lambda) \mid \lambda \in \Lambda_1 \} = \varphi(\Lambda_1). \quad (36)$$

To study the convex hull of \mathcal{F} and its boundary, we make use of the following two identities involving $P_c(V_L)$.

Lemma 3.8: For $x, z \in \mathbb{R}^n$ we have

$$P_c(x+z) = P_c(x) + \frac{\partial P_c}{\partial V_L}(x)z - [z]Y_{LL}z. \quad (37)$$

Proof: We write out the formula using (10) and use (11):

$$\begin{aligned} P_c(x+z) &= [x+z]Y_{LL}(V_L^* - x - z) = [x]Y_{LL}(V_L^* - x) \\ &\quad + [z]Y_{LL}(V_L^* - x) - [x]Y_{LL}z - [z]Y_{LL}z \\ &= P_c(x) + [Y_{LL}(V_L^* - x)]z - [x]Y_{LL}z - [z]Y_{LL}z \\ &= P_c(x) + \frac{\partial P_c}{\partial V_L}(x)z - [z]Y_{LL}z. \quad \square \end{aligned}$$

The matrix $h(\lambda)$ for $\lambda \in \Lambda$ is positive definite by definition, and therefore induces the vector norm

$$\|x\|_{h(\lambda)} := \sqrt{x^\top h(\lambda)x}. \quad (38)$$

This vector norm is related to $\lambda^\top P_c(x)$ for $\lambda \in \Lambda$ by the following lemma.

Lemma 3.9: Let $\lambda \in \Lambda$. For each $x \in \mathbb{R}^n$ we have

$$\lambda^\top P_c(x) = \|\varphi(\lambda)\|_{h(\lambda)}^2 - \|\varphi(\lambda) - x\|_{h(\lambda)}^2. \quad (39)$$

Moreover, we have

$$\lambda^\top P_c(x) \leq \|\varphi(\lambda)\|_{h(\lambda)}^2 = \lambda^\top P_c(\varphi(\lambda)), \quad (40)$$

with equality if and only if $x = \varphi(\lambda)$. Consequently, we have $P_c(x) = P_c(\varphi(\lambda))$ if and only if $x = \varphi(\lambda)$.

Proof: Using (10), (6), (35) and (28), we verify that

$$\begin{aligned} \lambda^\top P_c(x) &= \lambda^\top [x]Y_{LL}V_L^* - \lambda^\top [x]Y_{LL}x \\ &= x^\top [\lambda]I_L^* - x^\top [\lambda]Y_{LL}x \\ &= 2x^\top h(\lambda)\varphi(\lambda) - x^\top h(\lambda)x \\ &= \varphi(\lambda)^\top h(\lambda)\varphi(\lambda) - \varphi(\lambda)^\top h(\lambda)\varphi(\lambda) \\ &\quad + 2x^\top h(\lambda)\varphi(\lambda) - x^\top h(\lambda)x \\ &= \|\varphi(\lambda)\|_{h(\lambda)}^2 - \|\varphi(\lambda) - x\|_{h(\lambda)}^2. \end{aligned}$$

Eq. (40) follows from (39) since $\|\varphi(\lambda) - x\|_{h(\lambda)} \geq 0$, with equality if and only if $x = \varphi(\lambda)$. Thus, equality in (40) holds if and only if $x = \varphi(\lambda)$. Finally, if $P_c(x) = P_c(\varphi(\lambda))$, then $\lambda^\top P_c(x) = \lambda^\top P_c(\varphi(\lambda))$ therefore $x = \varphi(\lambda)$. \square

For a vector ν such that $\|\nu\|_1 = 1$ and a scalar s we define the closed half-space

$$H(\nu, s) := \{ y \mid \nu^\top y \leq s \},$$

which has as boundary the hyperplane

$$\partial H(\nu, s) = \{ y \mid \nu^\top y = s \}.$$

The vector ν is normal to the boundary of the half-space and points outwards.

Definition 3.10: A half-space $H(\nu, s)$ is said to *support* a set S if $S \subseteq H(\nu, s)$ and $\text{cl}(S) \cap \partial H(\nu, s)$ is nonempty. *I.e.*, for a given ν , s is the smallest number so that $S \subseteq H(\nu, s)$. A point in $\text{cl}(S) \cap \partial H(\nu, s)$ is a *point of support*.

If the half-space $H(\nu, s)$ supports \mathcal{F} and $\tilde{P}_c \in \mathcal{F}$ is a point of support, then $\nu^\top \tilde{P}_c$ maximizes the cost function $\nu^\top P_c$ for all $P_c \in \mathcal{F}$. For example, let $\nu = \frac{1}{n}\mathbb{1}$, then the corresponding supporting half-space is given by (25). The vector P_{\max} is the unique point of support, as was shown in Lemma 2.18. See also Figure 5, in which the dashed line corresponds to the boundary of this half-space.

In order to obtain a geometric description of the convex hull of \mathcal{F} we aim to apply the following proposition.

Proposition 3.11 (Cor. 11.5.1 of [19]): Let the set S be a subset of \mathbb{R}^n , then

$$\text{cl}(\text{conv}(S)) = \bigcap_{\text{support of } S} H(\nu, s)$$

where the intersection is taken over all half-spaces $H(\nu, s)$ which support S .

To apply Proposition 3.11 to \mathcal{F} , we identify all supporting half-spaces of the set \mathcal{F} . We will simultaneously identify the supporting half-spaces of the image of P_c , which is given by

$$\text{im } P_c := \{ y \mid P_c(x) = y, x \in \mathbb{R}^n \} \quad (41)$$

and satisfies the inclusion $\mathcal{F} \subseteq \text{im } P_c$.

Theorem 3.12: Let λ be a vector such that $\|\lambda\|_1 = 1$ and s be a scalar. A half-space $H(\lambda, s)$ supports \mathcal{F} if and only if $\lambda \in \Lambda_1$ and $s = \|\varphi(\lambda)\|_{h(\lambda)}^2$. The point $P_c(\varphi(\lambda)) \in \mathcal{F}$ is the unique point of support. Moreover, the supporting half-spaces of the set \mathcal{F} and the image of P_c coincide.

The proof of Theorem 3.12 can be found in Appendix C. To simplify notation, we define for $\lambda \in \Lambda$ the half-spaces

$$H_\lambda := H(\lambda, \|\varphi(\lambda)\|_{h(\lambda)}^2). \quad (42)$$

Theorem 3.12 states that H_λ for $\lambda \in \Lambda$ are all supporting half-spaces of \mathcal{F} . Proposition 3.11 therefore allows us to give a direct formula for the closure of the convex hull of \mathcal{F} .

Corollary 3.13: The closure of the convex hull of \mathcal{F} is the intersection of all half-spaces H_λ where $\lambda \in \Lambda$, and is equal to the closure of the convex hull of $\text{im } P_c$. *I.e.*,

$$\text{cl}(\text{conv}(\mathcal{F})) = \text{cl}(\text{conv}(\text{im } P_c)) = \bigcap_{\lambda \in \Lambda_1} H_\lambda. \quad (43)$$

Now that we have identified the closure of the convex hull of \mathcal{F} , we may also identify the boundary of this set.

Theorem 3.14: The map $\lambda \mapsto P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ is one-to-one and parametrizes the boundary of $\text{conv}(\mathcal{F})$. Moreover, the set $\text{conv}(\mathcal{F})$ is closed.

The proof of Theorem 3.14 can be found in Appendix C.

Corollary 3.15: The sets Λ_1 , $\partial\mathcal{D}$ and $\partial\text{conv}(\mathcal{F})$ are in one-to-one correspondence. In particular, P_c is a one-to-one map from $\partial\mathcal{D}$ to $\partial\text{conv}(\mathcal{F})$.

Proof: It follows from Theorem 3.7 that $\varphi(\lambda)$ is a one-to-one map from Λ_1 to $\partial\mathcal{D}$, and Theorem 3.14 states that $\lambda \rightarrow P_c(\varphi(\lambda))$ is a one-to-one map from Λ_1 to $\partial\text{conv}(\mathcal{F})$. Hence, P_c is a one-to-one map from $\partial\mathcal{D}$ to $\partial\text{conv}(\mathcal{F})$. \square

Corollary 3.15 states that a vector of power demands \tilde{P}_c that lies on the boundary the convex hull of \mathcal{F} corresponds uniquely to an operating point \tilde{V}_L which is long-term voltage semi-stable but not long-term voltage stable. The pair $(\tilde{P}_c, \tilde{V}_L)$ corresponds to a unique $\lambda \in \Lambda_1$, and the corresponding hyperplane ∂H_λ intersects $\text{conv}(\mathcal{F})$ only in the unique point of support \tilde{P}_c . Hence, ∂H_λ is the tangent plane at \tilde{P}_c of the boundary of $\text{conv}(\mathcal{F})$. This is observed in Figure 5 for $\tilde{P}_c = P_{\max}$, $\tilde{V}_L = \frac{1}{2}V_L^*$ and $\lambda = \frac{1}{n}\mathbb{1}$, and the same holds for all points on the boundary of $\text{conv}(\mathcal{F})$.

D. One-to-one correspondence between \mathcal{F} and $\text{cl}(\mathcal{D})$

In this section we use Corollary 3.15 to prove that P_c is a one-to-one mapping from $\text{cl}(\mathcal{D})$ to $\text{conv}(\mathcal{F})$, and that therefore \mathcal{F} is convex. This means each feasible power demand is uniquely associated to a long-term voltage semi-stable operating point. This operating point can be found by solving an initial value problem. The following lemma is instrumental in proving these results.

Lemma 3.16: Let $\tilde{V}_L \in \mathcal{D}$, define $\hat{P}_c := P_c(\tilde{V}_L) \in P_c(\mathcal{D})$ and let $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. There exists a unique path $\gamma : [0, 1] \rightarrow \mathcal{D}$ so that the convex combination of \hat{P}_c and \tilde{P}_c is described by

$$P_c(\gamma(\theta)) = \theta\tilde{P}_c + (1 - \theta)\hat{P}_c. \quad (44)$$

for $0 \leq \theta \leq 1$. The path γ solves the initial value problem

$$\dot{\gamma}(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(\theta)) \right)^{-1} (\tilde{P}_c - \hat{P}_c) \quad (45)$$

with initial value $\gamma(0) = \tilde{V}_L$. We have $\tilde{P}_c = P_c(\gamma(1))$.

The proof of Lemma 3.16 can be found in Appendix D.

Theorem 3.17 (M3): There is a one-to-one correspondence between the long-term voltage semi-stable operating points $\text{cl}(\mathcal{D})$ and the feasible power demands. *I.e.*, for each $\tilde{P}_c \in \mathcal{F}$ there exists a unique $\tilde{V}_L \in \text{cl}(\mathcal{D})$ which satisfies $\tilde{P}_c = P_c(\tilde{V}_L)$, implying that $\mathcal{F} = P_c(\text{cl}(\mathcal{D}))$. More explicitly, \tilde{V}_L is obtained by solving the initial value problem

$$\dot{\gamma}(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(\theta)) \right)^{-1} \tilde{P}_c \quad (46)$$

for $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ with initial value $\gamma(0) = V_L^*$ where the solution γ exists, is unique and satisfies $\gamma(1) = \tilde{V}_L$.

Proof: Note that $\tilde{P}_c \in \mathcal{F} \subseteq \text{conv}(\mathcal{F})$ and that $V_L^* \in \mathcal{D}$. Suppose $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. By taking $\tilde{V}_L = V_L^*$ and $\hat{P}_c = P_c(V_L^*) = 0$ in Lemma 3.16, there is a unique $\gamma : [0, 1] \rightarrow \mathcal{D}$ which solves (46) with $\gamma(0) = V_L^*$ and which satisfies $\tilde{P}_c = P_c(\gamma(1))$. Hence we take $\tilde{V}_L := \gamma(1) \in \mathcal{D}$. To show that there is a unique $\tilde{V}_L \in \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L)$, suppose that we have $\tilde{V}'_L \in \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}'_L)$. Then by Lemma 3.16 there is a unique $\gamma' : [0, 1] \rightarrow \mathcal{D}$ which solves

$$\dot{\gamma}'(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma'(\theta)) \right)^{-1} (P_c(V_L^*) - \tilde{P}_c) \quad (47)$$

with $\gamma'(0) = \tilde{V}'_L$, and satisfies $P_c(\gamma'(1)) = P_c(V_L^*) = 0$. Recall from Proposition 3.3 that $\gamma'(1) > 0$ since $\gamma'(1) \in \mathcal{D}$. Since $0 = P_c(\gamma'(1)) = [\gamma'(1)]Y_{LL}(V_L^* - \gamma'(1))$ and Y_{LL} is nonsingular, this implies that $\gamma'(1) = V_L^*$. But now note that $\hat{\gamma}(\theta) := \gamma'(1 - \theta) + V_L^* - \tilde{V}'_L$ is a solution to (46), since $P_c(V_L^*) = 0$ in (47) and since $\hat{\gamma}(0) = V_L^*$. Since (46) has a unique solution, it follows that $\hat{\gamma} = \gamma$, and in particular $\tilde{V}_L = \gamma(1) = \hat{\gamma}(1) = \tilde{V}'_L$, which proves that \tilde{V}_L is unique. Alternatively, if $\tilde{P}_c \in \partial\text{conv}(\mathcal{F})$ then by Corollary 3.15 there exists a unique $\tilde{V}_L \in \partial\mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L) \in P_c(\text{cl}(\mathcal{D}))$. It follows from Lemma 3.9 that there is no other operating point \tilde{V}'_L such that $\tilde{P}_c = P_c(\tilde{V}'_L)$. The operating point \tilde{V}_L is also obtained by the initial value problem (46), which follows from taking the limit $\tilde{P}_c \rightarrow \partial\mathcal{D}$ for $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. \square

In the context of the dynamical power grid (15) of Section II-C, Theorem 3.17 states that if an equilibrium to the system exists, then there always exists a semi-stable equilibrium $(\mathcal{I}^{\text{lines}}, V_L)$ where V_L is unique and lies in the set $\text{cl}(\mathcal{D})$.

The next theorem proves that the set \mathcal{F} of feasible power demands is closed and convex, and gives a geometric characterization of \mathcal{F} in terms of the closed half-spaces H_λ .

Theorem 3.18 (M4): The set \mathcal{F} of feasible power demands is closed and convex. Moreover, the set \mathcal{F} is the intersection of all half-spaces H_λ with $\lambda \in \Lambda_1$ (see (42)), and \mathcal{F} coincides with the image of P_c . *I.e.*,

$$\mathcal{F} = P_c(\text{cl}(\mathcal{D})) = \text{cl}(\text{conv}(\mathcal{F})) = \bigcap_{\lambda \in \Lambda_1} H_\lambda = \text{im } P_c. \quad (48)$$

Proof: We will first prove convexity. By definition we have $P_c(\text{cl}(\mathcal{D})) \subseteq \mathcal{F} \subseteq \text{conv}(\mathcal{F})$. Hence it suffices to show that $\text{conv}(\mathcal{F}) \subseteq P_c(\text{cl}(\mathcal{D}))$. Let $\tilde{P}_c \in \text{conv}(\mathcal{F})$. If $\tilde{P}_c \in \partial \text{conv}(\mathcal{F})$, then Corollary 3.15 implies that there exists $\tilde{V}_L \in \partial \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L) \in P_c(\text{cl}(\mathcal{D}))$. Alternatively, if $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$, then by Lemma 3.16 there exists a path $\gamma : [0, 1] \rightarrow \mathcal{D}$ such that $\gamma(0) = V_L^* \in \mathcal{D}$ and (44) holds. In particular, (44) implies that $\tilde{P}_c = P_c(\gamma(1)) \in P_c(\mathcal{D})$. Thus $\text{conv}(\mathcal{F}) \subseteq P_c(\text{cl}(\mathcal{D}))$, and thus \mathcal{F} is convex. Corollary 3.13, Theorem 3.14 and the convexity of \mathcal{F} further imply that

$$\bigcap_{\lambda \in \Lambda_1} H_\lambda = \text{cl}(\text{conv}(\mathcal{F})) = \text{conv}(\mathcal{F}) = \mathcal{F}. \quad (49)$$

Finally we show that $\mathcal{F} = \text{im } P_c$. Note that $\mathcal{F} \subseteq \text{im } P_c$ by definition. Corollary 3.13 proves that $\text{cl}(\text{conv}(\text{im } P_c)) = \text{cl}(\text{conv}(\mathcal{F}))$. We therefore have

$$\text{cl}(\text{conv}(\text{im } P_c)) = \text{cl}(\text{conv}(\mathcal{F})) = \mathcal{F} \subseteq \text{im } P_c.$$

Since $\text{im } P_c \subseteq \text{cl}(\text{conv}(\text{im } P_c))$, we have $\mathcal{F} = \text{im } P_c$. \square

Remark 3.19: Theorem 3.18 shows that the image of P_c coincides with \mathcal{F} . This means that, if nonpositive voltage potentials would be permitted, then any feasible power demand that is satisfied by nonpositive voltage potentials can also be satisfied by positive voltage potentials. Hence, from a theoretical standpoint, the restriction to positive voltage potentials does not make the set of feasible power demands more conservative.

Due to the convexity of \mathcal{F} , Corollary 3.15 implies that $\partial \mathcal{D}$ and $\partial \mathcal{F}$ in are one-to-one correspondence. Moreover, Lemma 3.9 implies that there are no vectors $x \notin \partial \mathcal{D}$ such that $P_c(x) \in \partial \mathcal{F}$. This implies the following corollary.

Corollary 3.20 (M2): For each \tilde{P}_c on the boundary of \mathcal{F} there exist a unique $\tilde{V}_L \in \mathbb{R}^n$ that satisfies (9). All such \tilde{V}_L satisfy $\tilde{V}_L > 0$ and form the boundary of \mathcal{D} . Hence, there is a one-to-one correspondence between $\partial \mathcal{D}$ and $\partial \mathcal{F}$.

Theorem 3.17 and Corollary 3.20 immediately imply that there is a one-to-one correspondence between the set \mathcal{D} of long-term voltage stable operating points and the power demands which are feasible under small perturbation, by which we mean that such a power demand \tilde{P}_c is feasible and does not lie on the boundary of \mathcal{F} (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F})$). Consequently, if a power demand is feasible under small perturbation, then there exists a unique long-term voltage stable operating point which satisfies the power flow equation.

Corollary 3.21: There is a one-to-one correspondence between the long-term voltage stable operating points \mathcal{D} and the feasible power demands under small perturbations $\text{int}(\mathcal{F})$.

IV. CONCLUSION OF PART I

In Part I of this paper we have studied the power flow feasibility of DC power grids with constant-power loads, and have presented a framework for the analysis of this feasibility problem. Specifically, we have presented a geometric characterization of the feasible power demands in terms of half-spaces. In addition, we have given a novel proof for the convexity of the set of feasible power demands. More importantly, we proved that there exists a one-to-one correspondence

between the feasible power demands and the long-term voltage semi-stable operating points. This shows that for each feasible power demand there exists a unique operating point which is long-term voltage semi-stable and satisfies the power flow equations. This operating point can be found by solving an initial value problem. The existence and uniqueness of this operating point proves that long-term (semi-)stability can be guaranteed for each feasible power demand. Furthermore, we showed that there exists a one-to-one correspondence between the feasible power demands under small perturbations and the long-term voltage stable operating points.

Our analysis is continued in Part II of this paper, in which we study necessary and sufficient conditions for feasibility, high-voltage operating points and sufficient conditions for power flow feasibility, among other things.

APPENDIX

A. Properties of Z-, M- and irreducible matrices

Proposition A.1: The Perron root and Perron vector of an irreducible Z-matrix A are continuous in the elements of A .

Proof: All eigenvalues and any eigenvector corresponding to a simple eigenvalue are continuous in the elements of the corresponding matrix (see [20, 3.1.2] and [20, 3.1.3]). \square

Proposition A.2 ([11, Thm. 5.8]): An irreducible M-matrix is singular if and only if its Perron root is zero, in which case its kernel is spanned by any Perron vector.

Proposition A.3: Consider the diagonal matrix $[d]$, $d \in \mathbb{R}^n$. The following statements hold:

- 1) If A is irreducible, then $A + [d]$ is irreducible;
- 2) If A is a Z-matrix, then $A + [d]$ is a Z-matrix;
- 3) If A is irreducible and $d > 0$, then $A[d]$ are $[d]A$ are irreducible;
- 4) If A is a Z-matrix and $d > 0$, then $A[d]$ are $[d]A$ are Z-matrices;
- 5) If A is an M-matrix and $d > 0$, then $A[d]$ are $[d]A$ are M-matrices;
- 6) If A is an irreducible M-matrix and $d \geq 0$, then $A + [d]$ is a nonsingular irreducible M-matrices;

Proof: Statements 1, 2, 3 and 4 follow directly from the definitions of a Z-matrix and irreducible matrix. Statement 5 is shown in [18]. Statement 6 follows from [21, Cor. 3.9] if A is singular, and is implied by [18, Thm. 1, A₃] if A is nonsingular. \square

Proposition A.4: The sum of two irreducible Z-matrices is an irreducible Z-matrix.

Proof: If $A_{[\alpha, \alpha^c]} \preceq 0$ and $B_{[\alpha, \alpha^c]} \preceq 0$ then $(A + B)_{[\alpha, \alpha^c]} \preceq 0$ for all nonempty $\alpha \subsetneq \mathbf{n}$. \square

B. Properties of Λ_1 and its closure

Recall from (29) and (33) that Λ (Λ_1) is the set of vectors λ such that $h(\lambda) = \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda])$ is positive definite (and $\|\lambda\|_1 = 1$).

Lemma B.5: The set Λ is an open convex cone.

Proof: The convex combination of positive definite matrices is again positive definite, and the set of all positive definite matrices is open. The result follows since $h(\lambda)$ is linear in λ . \square

Lemma B.6: The set Λ is contained in the positive orthant. I.e., $\lambda > 0$ for $\lambda \in \Lambda$.

Proof: Let λ be such that $h(\lambda)$ is positive definite. Recall that the matrix Y_{LL} is positive definite. A matrix is positive definite only if its diagonal elements are positive. The diagonal elements of Y_{LL} and $h(\lambda)$ are respectively given by $(Y_{LL})_{ii}$ and $\lambda_i(Y_{LL})_{ii}$, and therefore $(Y_{LL})_{ii} > 0$ and $\lambda_i(Y_{LL})_{ii} > 0$. This implies that $\lambda_i > 0$ for all i . \square

Lemma B.7: The set Λ_1 is a bounded convex set.

Proof: Let $\lambda \in \Lambda$. Since $\lambda > 0$ by Lemma B.6, it follows that $\|\lambda\|_1 = \lambda^\top \mathbf{1}$. Hence $\Lambda_1 = \Lambda \cap \{\lambda \mid \|\lambda\|_1 = 1\} = \Lambda \cap \{\lambda \mid \lambda^\top \mathbf{1} = 1\}$. The latter expression is an intersection of convex sets (see Lemma B.5). Hence Λ_1 is convex. The set $\{\lambda \mid \|\lambda\|_1 = 1\}$ is bounded and thus Λ_1 is bounded. \square

Lemma B.8: The closure of Λ_1 satisfies

$$\text{cl}(\Lambda_1) = \{\lambda \mid h(\lambda) \text{ is positive semi-definite, } \|\lambda\|_1 = 1\}.$$

Proof: Since Λ_1 is nonempty, this follows directly from linearity of h , and the fact that the positive semi-definite matrices form the closure of the positive definite matrices. \square

Lemma B.9: The set $\text{cl}(\Lambda_1)$ is contained in the positive orthant. Moreover, the matrix $h(\lambda)$ for $\lambda \in \text{cl}(\Lambda_1)$ is an irreducible M-matrix.

Proof: The vectors in Λ_1 are positive, and so the vectors in $\text{cl}(\Lambda_1)$ are nonnegative. To show that $\text{cl}(\Lambda_1)$ lies in the positive orthant, it suffices to show that if a vector $\lambda \in \partial\Lambda_1$ contains zeros, then Y_{LL} is not irreducible, which is a contradiction.

Suppose $\lambda \in \partial\Lambda_1$ such that $\lambda_{[\alpha]} = 0$ and $\lambda_{[\alpha^c]} > 0$ for some nonempty set $\alpha \subseteq \mathbf{n}$. Let w be a vector such that $w_{[\alpha^c]} = 0$ and $w_{[\alpha]}$ is arbitrary. We therefore have $[\lambda]w = 0$. Since $h(\lambda)$ is positive semi-definite, the following inequality holds for every vector v and scalar β :

$$0 \leq (v - \beta w)^\top h(\lambda)(v - \beta w) = (v - \beta w)^\top [\lambda]Y_{LL}(v - \beta w) = v^\top [\lambda]Y_{LL}(v - \beta w) = v^\top [\lambda]Y_{LL}v - \beta v^\top [\lambda]Y_{LL}w \quad (50)$$

If v is such that $v^\top [\lambda]Y_{LL}w \neq 0$, then (50) is violated when we take β such that $\beta v^\top [\lambda]Y_{LL}w$ is sufficiently large. It follows that $v^\top [\lambda]Y_{LL}w = 0$ for all v . This implies that

$$[\lambda]Y_{LL}w = 0. \quad (51)$$

The rows of (51) corresponding to $\alpha^c \subseteq \mathbf{n}$ satisfy

$$[\lambda_{[\alpha^c]}]((Y_{LL})_{[\alpha^c, \alpha]}w_{[\alpha]} + (Y_{LL})_{[\alpha^c, \alpha^c]}w_{[\alpha^c]}) = 0. \quad (52)$$

Recall that $\lambda_{[\alpha^c]} > 0$ and $w_{[\alpha^c]} = 0$, and thus (52) implies

$$(Y_{LL})_{[\alpha^c, \alpha]}w_{[\alpha]} = 0. \quad (53)$$

Since $w_{[\alpha]}$ is arbitrary, (53) should hold for all $w_{[\alpha]}$, and hence $(Y_{LL})_{[\alpha^c, \alpha]} = 0$. However, this contradicts the assumption that Y_{LL} is irreducible. We conclude that $\lambda > 0$.

Since Y_{LL} is an irreducible Z-matrix and $\lambda > 0$, Propositions A.3 and A.4 imply that $h(\lambda)$ is an irreducible Z-matrix. Since $h(\lambda)$ is positive semi-definite, its eigenvalues are real and nonnegative, and so $h(\lambda)$ is an M-matrix. \square

Lemma B.10: Let $\lambda \in \partial\Lambda_1$. For every $v \succeq 0$ in the kernel of $h(\lambda)$ we have $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$ for all $x \in \mathbb{R}^n$.

Proof: Since $\lambda \in \partial\Lambda_1$, the matrix $h(\lambda)$ is singular. By Lemma B.9, it is a singular irreducible M-matrix, and its Perron root is zero. The kernel of $h(\lambda)$ is spanned by any Perron vector, by Proposition A.2. This implies that v is a Perron vector and $v > 0$. Let x be any vector. By substituting (11), we note that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v$ is equivalent to

$$\begin{aligned} \lambda^\top \frac{\partial P_c}{\partial V_L}(x)v &= \lambda^\top [Y_{LL}V_L^*]v - \lambda^\top ([x]Y_{LL} + [Y_{LL}x])v \\ &= \lambda^\top [\mathcal{I}_L^*]v - 2x^\top h(\lambda)v = \lambda^\top [\mathcal{I}_L^*]v, \end{aligned} \quad (54)$$

where we used the fact that $h(\lambda)v = 0$. Since $v > 0$, $\lambda > 0$ and $\mathcal{I}_L^* \succeq 0$, it follows from (54) that $\lambda^\top [\mathcal{I}_L^*]v > 0$. We conclude that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$. \square

C. Proofs concerning Section III-C

Proof of Theorem 3.12: (\Leftarrow): The half-space $H(\lambda, s)$ with $\lambda \in \Lambda_1$ and $s = \|\varphi(\lambda)\|_{h(\lambda)}^2$ is given by

$$H(\lambda, s) = \left\{ y \mid \lambda^\top y \leq \|\varphi(\lambda)\|_{h(\lambda)}^2 \right\}.$$

Since $\lambda \in \Lambda_1$, Lemma 3.9 states that (40) holds for all $x \in \mathbb{R}^n$. This implies that $P_c(x) \in H(\lambda, s)$ for all $x \in \mathbb{R}^n$, and thus $\text{im } P_c \subseteq H(\lambda, s)$. To show that $P_c(\varphi(\lambda))$ is the unique point of support, we show that

$$\text{cl}(\text{im } P_c) \cap \partial H(\lambda, s) = \{P_c(\varphi(\lambda))\}. \quad (55)$$

Let $y \in \text{cl}(\text{im } P_c) \cap \partial H(\lambda, s)$, then there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \in \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} P_c(x_k) = y. \quad (56)$$

Since $y \in \partial H(\lambda, s)$, multiplying (56) by λ^\top yields

$$\lim_{k \rightarrow \infty} \lambda^\top P_c(x_k) = \lambda^\top y = s = \|\varphi(\lambda)\|_{h(\lambda)}^2. \quad (57)$$

It follows from rearranging (57) and applying (39) that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \left(\|\varphi(\lambda)\|_{h(\lambda)}^2 - \lambda^\top P_c(x_k) \right) \\ &= \lim_{k \rightarrow \infty} \|\varphi(\lambda) - x_k\|_{h(\lambda)}^2. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} x_k = \varphi(\lambda)$, and so (55) holds. This proves that $H(\lambda, s)$ supports $\text{im } P_c$, and that $P_c(\varphi(\lambda))$ is a point of support. The same is true for \mathcal{F} since $\mathcal{F} \subseteq \text{im } P_c$ and $P_c(\varphi(\lambda)) \in \mathcal{F}$.

(\Rightarrow): Let $x > 0$ be a vector. Let λ be such that $\|\lambda\|_1 = 1$ and $\lambda \notin \Lambda_1$, which means that $h(\lambda)$ is not positive definite. We will show that there exists a vector $v \geq 0$ such that $\lambda^\top P_c(x + tv)$ for scalars $t \geq 0$ is not bounded from above. In such a case we have $x + tv > 0$ for all $t \geq 0$. Hence the existence of such v implies that hyperplane $H(\lambda, s)$ does not contain \mathcal{F} for any scalar s , and the same holds for $\text{im } P_c$ since $\mathcal{F} \subseteq \text{im } P_c$. Lemma 3.8 tells us that

$$P_c(\hat{x} + tv) = P_c(\hat{x}) + t \frac{\partial P_c}{\partial V_L}(\hat{x})v - t^2 [v]Y_{LL}v. \quad (58)$$

We multiply (58) by λ^\top and use (28), which implies

$$\lambda^\top P_c(\hat{x} + tv) = \lambda^\top P_c(\hat{x}) + t \lambda^\top \frac{\partial P_c}{\partial V_L}(\hat{x})v - t^2 v^\top h(\lambda)v. \quad (59)$$

If $\lambda_i < 0$ for some i , then $e_i^\top h(\lambda)e_i = (Y_{LL})_{ii}\lambda_i < 0$. Hence, taking $v = e_i \not\geq 0$ in (59) describes a parabola in t which is not bounded from above. Thus $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$.

If $\lambda \geq 0$ and $\lambda \notin \text{cl}(\Lambda_1)$ then the matrix $h(\lambda)$ has a negative eigenvalue by Lemma B.8. Let r be the eigenvalue of $h(\lambda)$ with the smallest (*i.e.*, most negative) real part. Since $\lambda \geq 0$, it follows that $h(\lambda)$ is a Z-matrix. The matrix $h(\lambda)$ is block diagonal, where each block corresponds to an irreducible component of $h(\lambda)$. Let $h(\lambda)_{[\alpha, \alpha]}$ be the irreducible component that corresponds to the negative eigenvalue r . The matrix $h(\lambda)_{[\alpha, \alpha]}$ is an irreducible Z-matrix with Perron root r and Perron vector $w > 0$. Let v in (59) be such that $v_{[\alpha]} = w$ and $v_{[\alpha^c]} = 0$, then $v^\top h(\lambda)v = rw^\top w < 0$. It follows that (59) describes a parabola in t which is not bounded from above. Thus $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$.

Finally, suppose $\lambda \in \partial\Lambda_1$, which implies by Lemma B.9 that $\lambda > 0$ and that $h(\lambda)$ is an irreducible M-matrix. The matrix $h(\lambda)$ is singular since $\lambda \notin \Lambda_1$. Let $v > 0$ in (59) be a Perron vector of $h(\lambda)$. Proposition A.2 states that v spans the kernel of $h(\lambda)$, and so $v^\top h(\lambda)v = 0$. By Lemma B.10 we know that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$. This implies that (59) describes a half-line for $t \geq 0$ which is not bounded from above. Hence, $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$. \square

Proof of Theorem 3.14: The half-spaces H_λ for $\lambda \in \Lambda_1$ are all supporting half-spaces of $\text{conv}(\mathcal{F})$, which follows from (43) of Corollary 3.13. Theorem 3.12 proves that $P_c(\varphi(\lambda))$ is a point of support to H_λ , and that it is unique in the case of \mathcal{F} . Theorem 2.15 of [22] states that all boundary points of a convex set are a point of support associated to some supporting half-space. This implies that $P_c(\partial\mathcal{D}) \subseteq \partial\text{conv}(\mathcal{F})$. We prove equality by showing that there are no other points of support.

Let $y \in \text{cl}(\text{conv}(\mathcal{F}))$. Then there exists a sequence $\{y_k\}_{k \in \mathbb{N}} \in \text{conv}(\mathcal{F})$ such that $\lim_{k \rightarrow \infty} y_k = y$. This means that for $k \in \mathbb{N}$ there exists $x_k, z_k \in \mathbb{R}^n$ and scalars θ_k such that $y_k = \theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)$ and $0 < \theta_k < 1$. Suppose there exists $\tilde{\lambda} \in \Lambda_1$ so that $y \in \partial H_{\tilde{\lambda}}$. Hence, y is a point of support associated to $H_{\tilde{\lambda}}$. We define $s := \|\varphi(\tilde{\lambda})\|_{h(\tilde{\lambda})}^2$ and observe that

$$\begin{aligned} \lim_{k \rightarrow \infty} \tilde{\lambda}^\top (\theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)) \\ = \lim_{k \rightarrow \infty} \tilde{\lambda}^\top y_k = \tilde{\lambda}^\top y = s. \end{aligned}$$

Lemma 3.9 implies that for all k we have $\tilde{\lambda}^\top P_c(x_k) \leq s$, with equality if and only if $x_k = \varphi(\tilde{\lambda})$, and the same holds for z_k . This implies that

$$\tilde{\lambda}^\top (\theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)) \leq s. \quad (60)$$

In order to converge to equality in (60) as $k \rightarrow \infty$, we require that either $x_k \rightarrow \varphi(\tilde{\lambda})$ and $z_k \rightarrow \varphi(\tilde{\lambda})$, $x_k \rightarrow \varphi(\tilde{\lambda})$ and $\theta_k \rightarrow 1$, or $z_k \rightarrow \varphi(\tilde{\lambda})$ and $\theta_k \rightarrow 0$. In all cases it follows that $y = \lim_{k \rightarrow \infty} y_k = P_c(\varphi(\tilde{\lambda}))$. Hence $P_c(\varphi(\tilde{\lambda}))$ is the unique point of support.

Lemma 3.9 implies that if $P_c(\varphi(\lambda_1)) = P_c(\varphi(\lambda_2))$, then $\varphi(\lambda_1) = \varphi(\lambda_2)$. Hence the map $\lambda \rightarrow P_c(\varphi(\lambda))$ is a one-to-one correspondence between Λ_1 and $\partial\text{conv}(\mathcal{F})$, and $\lambda \rightarrow P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ parametrizes $\partial\text{conv}(\mathcal{F})$.

Note the inclusion

$$\partial\text{conv}(\mathcal{F}) = P_c(\partial\mathcal{D}) \subseteq \mathcal{F} \subseteq \text{conv}(\mathcal{F}),$$

which implies that $\text{conv}(\mathcal{F})$ is closed. \square

D. Proofs concerning Section III-D

Proof of Lemma 3.16: First we show that if a path $\gamma : [0, T] \rightarrow \mathcal{D}$ satisfies (45) with $\gamma(0) = \hat{V}_L \in \mathcal{D}$, then (44) holds. Indeed, note that the matrix $\frac{\partial P_c}{\partial V_L}(\gamma(\tau))$ is invertible for $0 \leq \tau \leq T$ since $\gamma(\tau) \in \mathcal{D}$, and note for $0 \leq \theta \leq T$ that by the fundamental theorem of calculus we have

$$P_c(\gamma(\theta)) = P_c(\gamma(0)) + \int_0^\theta \frac{\partial P_c}{\partial V_L}(\gamma(\tau)) \dot{\gamma}(\tau) d\tau. \quad (61)$$

Substitution of (45) in (61) yields

$$P_c(\gamma(\theta)) = P_c(\gamma(0)) + \int_0^\theta (\tilde{P}_c - \hat{P}_c) d\tau. \quad (62)$$

Eq. (44) follows from (62) since $P_c(\gamma(0)) = P_c(\hat{V}_L) = \hat{P}_c$.

To complete the proof it remains to show that a solution $\gamma(\theta) \in \mathcal{D}$ to (45) for $\theta \in [0, 1]$ exists and that this solution is unique. Let the map $\psi : \mathcal{D} \rightarrow \mathbb{R}^n$ be defined by

$$\psi(z) := \left(\frac{\partial P_c}{\partial V_L}(z) \right)^{-1} (\tilde{P}_c - \hat{P}_c).$$

The map $\psi(z)$ is continuously differentiable since $\frac{\partial P_c}{\partial V_L}(z)$ is invertible for $z \in \mathcal{D}$. Corollary 8.17 of [23] states that the initial value problem (45) has a unique solution $\gamma : (-\varepsilon, \varepsilon) \rightarrow B$ for some $\varepsilon > 0$, where B is an open neighborhood of \hat{V}_L which is contained in \mathcal{D} . Since $\psi(z)$ is continuous at all $z \in \mathcal{D}$, the solution γ can be extended to a maximal interval of existence. Indeed, by the Theorem 8.33 of [23] we extend γ so that either (i) $\gamma(\theta) \rightarrow \partial\mathcal{D}$, or (ii) $|\gamma(\theta)_i| \rightarrow \infty$ for some $i \in \mathbf{n}$, as $\theta \rightarrow \omega$ where $\omega \in \mathbb{R}_{>0} \cup \{+\infty\}$. We will treat cases (i) and (ii) separately.

Case (i): Let $x \in \partial\mathcal{D}$ such that $\gamma(\theta) \rightarrow x$ as $\theta \rightarrow \omega$ and let $y := P_c(x)$. By continuity of $P_c(V_L)$ it follows that $P_c(\gamma(\theta)) \rightarrow y$ as $\theta \rightarrow \omega$. Since $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta < \omega$, the first part of this proof showed that (44) holds for $0 \leq \theta < \omega$. Suppose $\omega = 1$, then taking the limit $\theta \rightarrow \omega$ in (44) implies that $\tilde{P}_c = P_c(\gamma(\omega)) = y$, which lies on the boundary of \mathcal{F} . This contradicts the fact that $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. Suppose $1 > \omega$, then (44) implies that $P_c(\gamma(\omega)) = y$ is a convex combination of \hat{P}_c and \tilde{P}_c . Since $\hat{V}_L \notin \partial\mathcal{D}$ it follows from Theorem 3.14 that $\hat{P}_c \notin \partial\text{conv}(\mathcal{F})$. Since $\hat{P}_c \in \mathcal{F}$ we therefore have $\hat{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. Let $\lambda \in \Lambda_1$ such that $P_c(\varphi(\lambda)) = y$, which exists by Theorem 3.14, and define $s := \|\varphi(\lambda)\|_{h(\lambda)}^2 = \lambda^\top P_c(\varphi(\lambda)) = \lambda^\top y$. Note that $\lambda^\top \hat{P}_c < s$ and $\lambda^\top \tilde{P}_c < s$ since $\hat{P}_c, \tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. But since y is a convex combination of \hat{P}_c and \tilde{P}_c , this would imply that $\lambda^\top y < s$, which is a contradiction. We conclude that $1 < \omega$, and in particular $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta \leq 1$.

Case (ii): We will show that $\omega = +\infty$ and that $P_c(\gamma(\theta))$ describes a half-line for $0 \leq \theta < \infty$. Let $\lambda \in \Lambda_1$. Note that $|\gamma(\theta)_i| \rightarrow \infty$ implies that also $|\varphi(\lambda)_i - \gamma(\theta)_i| \rightarrow \infty$. Therefore also $\|\varphi(\lambda) - \gamma(\theta)\|_{h(\lambda)} \rightarrow \infty$. It follows from Lemma 3.9 that $\lambda^\top P_c(\gamma(\theta)) \rightarrow -\infty$. This holds for all $\lambda \in \Lambda_1$ and so $P_c(\gamma(\theta))$ does not intersect the boundary of \mathcal{F} for $0 \leq \theta < \omega$. The first part of this proof showed that (44) holds for $0 \leq \theta < \omega$, which describes a half-line in θ . Since $\lambda > 0$ by Lemma B.6, it follows from $\lambda^\top P_c(\gamma(\theta)) \rightarrow -\infty$ that $P_c(\gamma(\theta))_j \rightarrow -\infty$ for some $j \in \mathbf{n}$. As a result, (44) implies that $\omega = +\infty$. In particular it follows that \tilde{P}_c lies on the half-line and that $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta \leq 1$.

To show uniqueness, we note again that $\psi(z)$ is continuously differentiable. Corollary 8.17 of [23] states that (45) has a unique solution in an open neighborhood around any given initial value in \mathcal{D} . Taking any point $\gamma(\theta)$ with $0 \leq \theta \leq 1$ as an initial value shows that the solution γ is unique at each point, and hence is unique in \mathcal{D} .

Since (44) holds for $0 \leq \theta \leq 1$, (44) implies that $\tilde{P}_c = P_c(\gamma(1)) \in P_c(\mathcal{D})$. \square

REFERENCES

- [1] T. Van Cutsem and C. Vournas, *Voltage stability of electric power systems*. Springer Science & Business Media, 2008.
- [2] U. Eminoglu and M. H. Hocaoglu, "A new power flow method for radial distribution systems including voltage dependent load models," *Electric Power Systems Research*, vol. 76, no. 1, pp. 106 – 114, 2005.
- [3] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage collapse in complex power grids," *Nature Communications*, vol. 7, no. 10790, 2016.
- [4] A. S. Matveev, J. E. Machado, R. Ortega, J. Schiffer, and A. Pyrkyn, "A tool for analysis of existence of equilibria and voltage stability in power systems with constant power loads," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4726–4740, 2020.
- [5] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proceedings of the National Academy of Sciences*, vol. 110, no. 6, pp. 2005–2010, 2013.
- [6] S. Bolognani and S. Zampieri, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 163–172, 2015.
- [7] N. Barabanov, R. Ortega, R. Griñó, and B. Polyak, "On existence and stability of equilibria of linear time-invariant systems with constant power loads," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 1, pp. 114–121, Jan 2016.
- [8] A. Dymarsky, "On the convexity of image of a multidimensional quadratic map," *arXiv preprint arXiv:1410.2254v2*, 2018.
- [9] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "A solvability condition for reactive power flow," in *2015 54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 2013–2017.
- [10] A. Dymarsky and K. Turitsyn, "Convexity of solvability set of power distribution networks," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 222–227, Jan 2019.
- [11] M. Fiedler, *Special matrices and their applications in numerical mathematics*. Kluwer Academic Publishers, 1986.
- [12] A. van der Schaft, "Characterization and partial synthesis of the behavior of resistive circuits at their terminals," *Systems & Control Letters*, vol. 59, no. 7, pp. 423 – 428, 2010.
- [13] A. van der Schaft, "The flow equations of resistive electrical networks," in *Interpolation and Realization Theory with Applications to Control Theory: In Honor of Joe Ball*. Springer International Publishing, 2019, pp. 329–341.
- [14] F. Dörfler and F. Bullo, "Novel insights into lossless AC and DC power flow," in *2013 IEEE Power Energy Society General Meeting*, 2013, pp. 1–5.
- [15] W. Rudin *et al.*, *Principles of mathematical analysis*. McGraw-Hill New York, 1964, vol. 3.

- [16] P. Kundur, J. Paserba, V. Ajjarapu, G. Andersson, A. Bose, C. Canizares, N. Hatziargyriou, D. Hill, A. Stankovic, C. Taylor, T. Van Cutsem, and V. Vittal, "Definition and classification of power system stability IEEE/CIGRE joint task force on stability terms and definitions," *IEEE Transactions on Power Systems*, vol. 19, no. 3, pp. 1387–1401, Aug 2004.
- [17] F. Zhang, *The Schur complement and its applications*. Springer Science & Business Media, 2006, vol. 4.
- [18] R. Plemmons, "M-matrix characterizations I - nonsingular M-matrices," *Linear Algebra and its Applications*, vol. 18, no. 2, pp. 175 – 188, 1977.
- [19] R. T. Rockafellar, *Convex analysis*. Princeton University Press, 1970, vol. 28.
- [20] J. M. Ortega, *Numerical analysis: a second course*. SIAM, 1990.
- [21] W. Li, "Characterizations of singular irreducible m-matrices," *Linear and Multilinear Algebra*, vol. 38, no. 3, pp. 241–247, 1995.
- [22] F. A. Valentine, *Convex sets*. McGraw-Hill New York, 1964.
- [23] W. G. Kelley and A. C. Peterson, *The theory of differential equations: classical and qualitative*. Springer Science & Business Media, 2010.

Mark Jeeninga received his master degree in Mathematics at the University of Groningen, The Netherlands in 2015. He has obtained his PhD degree at the same university in 2021, and is currently a postdoctoral researcher at the Polytechnic University of Turin. His research interests include power flow analysis, smart grids, and matrix theory.

Claudio De Persis is a Professor with the Engineering and Technology Institute, Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands. He received the Laurea degree in electronic engineering in 1996 and the Ph.D. degree in system engineering in 2000 both from the University of Rome La Sapienza, Rome, Italy. Before joining the University of Groningen, he held faculty positions with the Department of Mechanical Automation and Mechatronics, University of Twente and the Department of Computer, Control, and Management Engineering, University of Rome La Sapienza. He was a Research Associate with the Department of Systems Science and Mathematics, Washington University, St. Louis, MO, USA, from 2000–2001, and with the Department of Electrical Engineering, Yale University, New Haven, CT, USA, from 2001–2002. His main research interest includes control theory, and his recent research focuses on learning for control, networked cyber-physical systems, smart grids, and resilient control. Dr. Persis was an Editor of the International Journal of Robust and Nonlinear Control (2006–2013), an Associate Editor of the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY (2010–2015), and of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL (2012–2015). He is currently an Associate Editor of Automatica (2013–present) and of IEEE CONTROL SYSTEMS LETTERS (2017–present).

Arjan van der Schaft received the undergraduate and PhD degrees in mathematics from the University of Groningen, the Netherlands. In 1982 he joined the Department of Applied Mathematics, University of Twente, where he was appointed as full professor in Mathematical Systems and Control Theory in 2000. In September 2005 he returned to his Alma Mater as professor in Mathematics.

Arjan van der Schaft is Fellow of the Institute of Electrical and Electronics Engineers (IEEE), Fellow of the International Federation of Automatic Control (IFAC), and was the 2013 recipient of the 3-yearly awarded Certificate of Excellent Achievements of the IFAC Technical Committee on Nonlinear Systems. He was Invited Speaker at the International Congress of Mathematicians, Madrid, 2006.

Books authored by him include Variational and Hamiltonian Control Systems (1987, with P.E. Crouch), Nonlinear Dynamical Control Systems (1990, 2016, with H. Nijmeijer), L2-Gain and Passivity Techniques in Nonlinear Control (1996, 2000, 2017), An Introduction to Hybrid Dynamical Systems (2000, with J.M. Schumacher), and Port-Hamiltonian Systems: An Introductory Overview (2014, with D. Jeltsema)