Braess' paradox for power flow feasibility and parametric uncertainties in DC power grids with constant-power loads*

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Abstract

This paper studies the power flow feasibility of DC power grids with constant-power loads. We introduce and motivate the concept of Braess' paradox for power flow feasibility, and show that this phenomenon can occur in most practical power grids with at least two source nodes. To combat Braess' paradox we consider bounded parametric uncertainties in the power lines, and give a simple but conservative sufficient condition to guarantee that the power flow is feasible under these parametric uncertainties. We refer to this property as strong feasibility.

Keywords: Braess' paradox, DC power grids, Constant-power loads, Parametric uncertainties

1. Introduction

Braess' paradox is a phenomenon in traffic flow networks, first observed in [1], described by the counterintuitive observation that adding a road (or improving the capacity of a road) does not necessarily improve the traffic flow. This phenomenon is a prevalent property of such networks [2]. Also in the case of power grids, several adverse effects have been observed that are due to the addition of a line. Although these reported phenomena are all referred to as Braess' paradox, a overarching definition of this paradox is missing. We give a brief overview.

The papers [3, 4] report several physical examples of two-port DC circuits for which the current flow exhibits behavior analogous to Braess' paradox in traffic flow networks. Analogous to the original example in [1], the current injections at the two ports are taken constant, and the lines are a mixture of lines with a linear flow and a nonlinear flow. The latter is achieved by Zener diodes, or by resistive lines with an additional constant voltage drop. In these examples it is shown that the addition of a line can lead to an increase of the current flows in all lines, which goes against the intuition that adding a line allows for improved flow in the network. This is the phenomenon known as Braess' paradox. Furthermore, the power consumption of these circuits is increased by the addition of the line. In contrast to [3, 4], the increased consumption that results from adding a line has also been referred to as Braess' paradox [5]. These phenomena have also been studied in [6, 7], which considers the effects of adding a line (or an increase of the conductance of a line) in DC circuits with voltage-controlled or current-controlled nodes.

It is noted that [1, 3, 4] show the existence Braess' paradox by means of Wheatstone bridges using components with nonlinear flow, that cannot be captured in standard models for AC and DC power grids. Nevertheless, similar phenomena have also been observed for the power flow in both AC and DC power grids. The papers [8, 9] provide examples of power grids where nodal power injections are fixed. The example in [8] is of a DC power grid, and observes that the maximal transmitted power of any line could both increase or decrease when a new line is added. The example in [9] is of an AC power grid, and observes that introducing a new line or increasing the conductance of a line can destabilize the power grid. These phenomena are also referred to as Braess' paradox, but are phrased in terms of the quality of the power flow and the stability of the system rather than the current flow.

In this paper we study the power flow of DC power grids with fixed voltage sources and constant-power loads. Such loads may appear in practical power grids, and are known to destabilize the power grid due to their negative impedance characteristic [10]. In these power grids it may occur that that the sources cannot satisfy the power demands of the loads, in which case we say that the power flow is unfeasible. The feasibility of the power flow is important for their long-term operation, since sustained unfeasible power flow may lead to unintended behavior system such as voltage oscillations, voltage collapse, and blackouts [11, 12]. This feasibility problem is a classical problem in the literature [13, 14, 15] and has gained more attention over the past decade [16, 17, 18, 19]. A full characterization of this feasibility problem has been presented in [20, 21].

The aim of the present paper is to study how the feasibility of the power flow in these systems is affected by changes in the line conductances. This study is primarily motivated by [9], where it was observed that increasing the conductance of a line in an AC power grid leads to voltage oscillations, which can be attributed to unfeasibility of the power flow after this increase.

The contributions of the present paper are of a theoretical nature, and are divided into two parts. The first part of this pa-

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per shows that an analogue of Braess' paradox can also occur in DC power grids with constant-power loads, and that this may occur for most practical power grids with multiple sources. The prevalent occurrence of this phenomenon motivates the second part of this paper, in which we study power grids where the exact line conductances are not known, and only bounds on line conductances are available. For such power grids we formulate the concept of strong feasibility and give a sufficient condition which guarantees that the power flow in the power grid is strongly feasible.

The structure of this paper is as follows. In Section 2 we state the model for DC power grids with constant-power loads at steady state. In Section 3 we formulate the Braess' paradox of power flow feasibility, and show that Braess' paradox can occur in most practical power grids. Section 4 studies the power flow feasibility of power grids with bounded uncertainties in the lines. Section 5 concludes the paper.

Notation

For a vector $x = \begin{pmatrix} x_1 & \cdots & x_k \end{pmatrix}^\top$ we denote

$$[x] := \operatorname{diag}(x_1, \ldots, x_k).$$

We let $\mathbb{1}$ and $\mathbb{0}$ denote the all-ones and all-zeros vector, respectively, and let I denote the identity matrix. We let their dimensions follow from their context. We let e_i denote the i-th column of I. All vector and matrix inequalities are taken to be elementwise. We write $x \lneq y$ if $x \leq y$ and $x \neq y$.

2. The power grid model

Throughout this paper we study DC power grids at steady-state, and model such systems by a resistive circuit. We model a power grid consisting of n load nodes and m source nodes as follows. If distinct nodes i and j are connected by a line, we let $G_{ij} = G_{ji} > 0$ denote the conductance of this line. If the nodes are not connected by a line we put $G_{ij} = G_{ji} = 0$. The Kirchhoff matrix $Y \in \mathbb{R}^{(n+m)\times(n+m)}$ associated to the lines in the grid is defined by

$$Y_{ij} := \begin{cases} \sum_k G_{ki} & \text{if } i = j \\ -G_{ij} & \text{if } i \neq j \end{cases}. \tag{1}$$

The voltage potentials and injected currents at the loads are collected in the vectors $V \in \mathbb{R}^{n+m}$ and $\mathcal{I} \in \mathbb{R}^{n+m}$, respectively. The quantities V, $\mathcal{I}$ and Y are partioned as

$$V = \begin{pmatrix} V_L \\ V_S \end{pmatrix}; \quad \mathcal{I} = \begin{pmatrix} \mathcal{I}_L \\ \mathcal{I}_S \end{pmatrix}; \quad Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix},$$

according to whether nodes are loads (L) or sources (S). We assume that the nodes in the power grid are connected, which means that $\mathbb{1}$ spans the kernel of Y, and that the principal submatrices Y_{LL} and Y_{SS} are positive definite. Due to Kirchhoff's and Ohm's laws we have $\mathcal{I} = YV$. We define the open-circuit voltages $V_L^* > \mathbb{0}$ to be the unique vector of voltage potentials at the loads such that $\mathcal{I}_L = \mathbb{0}$, which satisfies

$$V_L^* := -{Y_{LL}}^{-1} Y_{LS} V_S \tag{2}$$

(*e.g.*, see [22]). The power injected at the nodes is given by $P = [V]\mathcal{I}$. Since $\mathcal{I} = YV$ and due to (2) we have

$$P_L = [V_L](Y_{LL} V_L + Y_{LS} V_S) = [V_L] Y_{LL} (V_L - V_L^*) \tag{3a}$$
$$P_S = [V_S](Y_{SS} V_S + Y_{SL} V_L). \tag{3b}$$

The total dissipated power in the lines is given by

$$R(V_L, V_S) := V^\top Y V = \sum_{i,j:\, i>j} G_{ij}(V_i - V_j)^2, \tag{4}$$

whereas the total power injected at the nodes is $\mathbb{1}^\top P = V^\top \mathcal{I}$. Since $\mathcal{I} = YV$, it is observed that the total dissipated power in the lines equals the total power injected at the loads:

$$R(V_L, V_S) = V^\top Y V = \mathbb{1}^\top P = \mathbb{1}^\top P_L + \mathbb{1}^\top P_S. \tag{5}$$

We assume that all loads demand a constant power. Since we study the power grid at steady state, this is to say that we want to choose V_L such that all power demands are satisfied. The constant power demands are collected in the vector $P_c \in \mathbb{R}^n$. The question if such a V_L exists for a given P_c gives rise to the DC power flow feasibility problem:

Definition 2.1. *Given a power grid with Kirchhoff matrix Y, source voltages $V_S > \mathbb{0}$ and constant power demands P_c, we say that the power flow (of the power grid) is* feasible *if there exists a vector $V_L > \mathbb{0}$ of load voltages such that*

$$P_c = -P_L = [V_L] Y_{LL} (V_L^* - V_L).$$

Put differently, feasibility of a power grid means that the constant power demands at the loads can be satisfied at steady state. It is noted from (3a) that if Y_{LL} is reducible (or equivalently, block-diagonal), then power flow feasibility can be analyzed for each reducible component separately. Hence without loss of generality we assume that Y_{LL} is irreducible, which is equivalent to saying that the subgraph induced by the load nodes is connected [23, Thm. 3.6.a].

The scalar $-\mathbb{1}^\top P_L$ represents the total amount of power that is drained by the loads. Intuitively, the total amount of power that can be drained by the loads is bounded from above. Recall that the power flow is feasible we have that $P_c = -P_L$. Hence this means that also $\mathbb{1}^\top P_c$, the total power demand of the load nodes, is bounded from above whenever the power flow is feasible. The *maximizing power demand*, given by

$$P_{\max} := -\tfrac{1}{4}[V_L^*] Y_{LS} V_S, \tag{6}$$

is the unique vector of constant power demands for which the power flow is feasible and the total power demand is maximized:

Proposition 2.2 ([20, Lem. 2.17])**.** *Consider a power grid with Y and $V_S > \mathbb{0}$ given. If $P_c \in \mathbb{R}^n$ is a vector of power demands such that the power flow is feasible, then*

$$\mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max}, \tag{7}$$

with equality if and only if $P_c = P_{\max}$. Put differently, the quantity $\mathbb{1}^\top P_{\max}$ is the maximal total power demand that can be satisfied by the power grid. The unique voltage potentials corresponding to $P_{\max}$ are $V_L = \frac{1}{2} V_L^*$.

Note that (7) is a necessary condition for power flow feasibility.

3. Braess' paradox in DC power grids with constant-power loads

The classical formulation of Braess' paradox from [1, 3, 4] is that, after increasing line conductances or adding a line in a DC circuit, the quality of the current flow in the power grid becomes measurably worse. In this paper however we follow a stronger formulation of Braess' paradox, inspired by [9], by studying how adding lines or increasing conductances can destabilize the power grid, and in particular, lose power flow feasibility. We formalize this as follows:

Definition 3.1. Braess' paradox for power flow feasibility *is the phenomenon that adding a line or increasing a line conductance in a power grid destroys the feasibility of the power flow and destabilizes the power grid. We say that Braess' paradox for power flow feasibility can occur in a power grid if there exists a vector of power demands P_c such that the power flow is feasible and becomes unfeasible after increasing a line conductance or adding a line.*

In the remainder of this paper we refer to Definition 3.1 simply as the Braess' paradox, for the sake of brevity.

3.1. A sufficient condition for the occurrence Braess' paradox

In this section we study the maximal total power demand $\mathbb{1}^\top P_{\max}$, and give a formal proof for the occurrence of Braess' paradox by analyzing this quantity. We start by showing that the maximum total power that can be transported by the power grid never increases when the conductance of a line between two load nodes increases.

Theorem 3.2. *The maximal total power demand $\mathbb{1}^\top P_{\max}$ is nonincreasing as a function of the conductances of lines between loads.*

Proof. We consider a power grid with Kirchhoff matrix Y, fixed source voltages $V_S > \mathbb{0}$ and fixed load voltages $V_L > \mathbb{0}$. It follows from (5) that

$$-\mathbb{1}^\top P_L = \mathbb{1}^\top P_S - R(V_L, V_S), \tag{8}$$

which states that the total power that is drained by the loads is equal to the difference of the power supplied by the generators and the power dissipated in the lines. Note from (4) that $R(V_L, V_S)$ is nondecreasing in all conductances G_{ij}. By (3b), the quantity $\mathbb{1}^\top P_S$ is independent of the conductances of lines between loads, since such conductances only appear in the submatrix Y_{LL}. It therefore follows from (8) that $-\mathbb{1}^\top P_L$ is nonincreasing in the conductances of lines between loads. We now consider the situation where we increase G_{ij}, where i and j are loads. We let $P_{\max}$ be the maximizing power demand *before* this event. Similarly, we let $\widehat{P}_{\max}$ be the maximizing power demand *after* this event and let $\widehat{V}_L$ be the corresponding vector of voltage potentials (see Proposition 2.2). We now choose the fixed load voltages to be $V_L = \widehat{V}_L$, which means that $\widehat{P}_{\max} = -P_L$ after the event, and hence $\mathbb{1}^\top \widehat{P}_{\max} \le -\mathbb{1}^\top P_L$. Since $-\mathbb{1}^\top P_L$ is nonincreasing in G_{ij}, we know that $\mathbb{1}^\top \widehat{P}_{\max} \le -\mathbb{1}^\top P_L$ before the event. We let P_c be the vector of power demands at the loads so that $P_c = -P_L$ before the event, meaning that $\mathbb{1}^\top \widehat{P}_{\max} \le \mathbb{1}^\top P_c$. By Proposition 2.2 we have that $\mathbb{1}^\top P_c \le \mathbb{1}^\top P_{\max}$ before the event. We conclude that $\mathbb{1}^\top \widehat{P}_{\max} \le \mathbb{1}^\top P_{\max}$. □

In the proof of Theorem 3.2 we show for fixed voltage potentials that $-\mathbb{1}^\top P_L$, the total power drained by the nodes, is nonincreasing in the conductances of lines between loads, which is due to the fact that increasing such a conductance will increase the dissipation in the lines (see (8)).

Theorem 3.2 shows that an increase of a line conductance can make the maximal total power demand decrease. This means that the power flow in a power grid can become unfeasible after such an increase, in particular when the total power demand $\mathbb{1}^\top P_c$ is close to its maximum $\mathbb{1}^\top P_{\max}$, such as when $P_c = P_{\max}$. Hence, following Definition 3.1, we have that Braess' paradox occurs. We formalize this in the following theorem, which presents a sufficient condition for the existence of a pair of loads for which Braess' paradox occurs when the line between them is altered (or added). This theorem generalizes Theorem 3.2, by identifying when $\mathbb{1}^\top P_{\max}$ is strictly decreasing.

Theorem 3.3. *Consider a power grid with Kirchhoff matrix Y, source voltages $V_S > \mathbb{0}$ and at least two loads and at least two sources. Let P_c be a vector of constant power demands such that the power flow is feasible. If there exist (distinct) load nodes i and j such that the open-circuit voltages* (2) *satisfy $(V_L^*)_i \neq (V_L^*)_j$, and if P_c satisfies*

$$\mathbb{1}^\top P_c > \mathbb{1}^\top P_{\max} - \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{(e_i - e_j)^\top {Y_{LL}}^{-1}(e_i - e_j)}, \tag{9}$$

then there exists a scalar $c > 0$ such that the power flow becomes unfeasible after increasing G_{ij} by c, either through increasing the conductance of the line between loads i and j, or adding a new line between loads i to j. In particular, if we consider $P_c = P_{\max}$, then the power flow becomes unfeasible for any increase of G_{ij}.

Proof. Increasing G_{ij} by a positive scalar c corresponds to adding the matrix $\Delta := c(e_i - e_j)(e_i - e_j)^\top$ to Y_{LL}. By substituting (2) in (6) we observe that

$$\mathbb{1}^\top P_{\max} = -\tfrac{1}{4}(V_L^*)^\top Y_{LS} V_S = \tfrac{1}{4} V_S^\top Y_{SL} {Y_{LL}}^{-1} Y_{LS} V_S.$$

We let $\widehat{P}_{\max}$ be the maximizing power demand of the power grid after the increase of G_{ij} by c. We have

$$\mathbb{1}^\top \widehat{P}_{\max} = \tfrac{1}{4} V_S^\top Y_{SL} (Y_{LL} + \Delta)^{-1} Y_{LS} V_S.$$

The difference between the two total power demands is $d := \mathbb{1}^\top P_{\max} - \mathbb{1}^\top \widehat{P}_{\max}$. By Lemma A.1 we have

$$\begin{aligned} d &= \tfrac{1}{4} c \frac{(V_S^\top Y_{SL} {Y_{LL}}^{-1}(e_i - e_j))^2}{1 + c(e_i - e_j)^\top {Y_{LL}}^{-1}(e_i - e_j)} \\ &= c \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{1 + c(e_i - e_j)^\top {Y_{LL}}^{-1}(e_i - e_j)}, \end{aligned} \tag{10}$$

where we substituted (2). Since Y_{LL} is positive definite and $c > 0$ it follows that the denominator of (10) is positive. Since

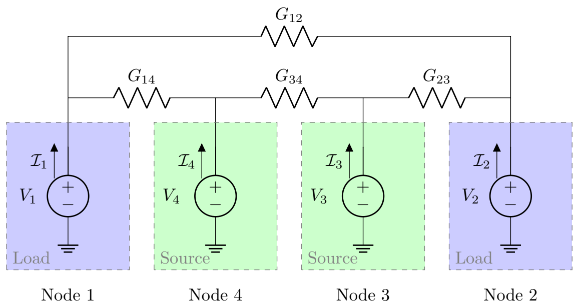

Figure 1: A schematic depiction of a DC power grid with two loads and two sources ($n = 2$, $m = 2$) for which Braess' paradox can occur.

$(V_L^*)_i \neq (V_L^*)_j$ we have $d > 0$, and therefore $\mathbb{1}^\top \widehat{P}_{\max} < \mathbb{1}^\top P_{\max}$. Since $\widehat{P}_{\max}$ is the maximizing power demand corresponding the power grid after increasing G_{ij}, we have by Proposition 2.2 that $\mathbb{1}^\top \widehat{P}_{\mathrm{c}} < \mathbb{1}^\top \widehat{P}_{\max}$ for all power demands $\widehat{P}_{\mathrm{c}} \neq \widehat{P}_{\max}$ for which the power flow of the *altered* power grid is feasible. Since $\mathbb{1}^\top \widehat{P}_{\max} < \mathbb{1}^\top P_{\max}$, this implies that the power flow is no longer feasible after increasing G_{ij} when $P_{\mathrm{c}} = P_{\max}$. By taking $c \to \infty$ in (10), (*i.e.*, by shorting the line between i and j), we observe

$$d \nearrow \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{(e_i - e_j)^\top {Y_{LL}}^{-1}(e_i - e_j)}.$$

And so, for every P_{c} such that the power flow is feasible and such that (9) holds we can take c large enough such that $\mathbb{1}^\top \widehat{P}_{\max} < \mathbb{1}^\top P_{\mathrm{c}}$, which implies by Proposition 2.2 that the power flow becomes unfeasible. Finally, when the power grid has exactly one source node, we have due to $Y\mathbb{1} = \mathbb{0}$ that

$$\mathbb{0} = Y_{LL}\mathbb{1} + Y_{LS}\mathbb{1} = Y_{LL}\mathbb{1} + Y_{LS},$$

from which it follows that

$$V_L^* = -\tfrac{1}{2} Y_{LL}^{-1} Y_{LS} V_S = \tfrac{1}{2}\mathbb{1} V_S,$$

which indicates that all open-circuit voltages are the same and $P_{\max}$ never changes. □

Interestingly, we observe that the denominator of the fraction (9) corresponds to the effective resistance between nodes i and j with respect to the loopy Laplacian Y_{LL} (*e.g.*, see [24]), whereas the quantities $\frac{1}{2}V_L^*$ in the numerator are the voltage potentials corresponding to $P_{\max}$.

Theorem 3.3 tells us that if the open-circuit voltages are not all equal (*i.e.*, $V_L^* \notin \operatorname{span}\{\mathbb{1}\}$), then Braess' paradox can occur when the power demands are close to the maximizing power demands $P_{\max}$. We illustrate this by the following example.

Example 3.4. *Consider the power grid with two loads and two sources, as depicted in Figure 1, where* $V_S = \begin{pmatrix}1 & 3\end{pmatrix}^\top$, $G_{12} = 0.3$, $G_{14} = 1$, $G_{23} = 5$ *and* $G_{34} = 1$. *The corresponding open-circuit voltages are* $V_L^* = \frac{1}{6.8}\begin{pmatrix}17.4 & 7.4\end{pmatrix}^\top$, *and are not a multiple of* $\mathbb{1}$. *The blue area in Figure 2 depicts the set of all vectors* P_{c} *such that the power flow is feasible. Since* $(V_L^*)_1 \neq (V_L^*)_2$, *Theorem 3.3 states that we will observe Braess' paradox if we increase the line between load* 1 *and load* 2. *We increase the conductance of this line by* 0.7. *This results in the green area in Figure 2 corresponding to the set* $\widehat{F}$ *of* P_{c} *such that the power flow is feasible. It is observed that the blue area is not contained in the green area. Hence there are vectors of power demands for which the power flow has become unfeasible after increasing the conductance of the line. In particular we see that* P_{max} *is no longer feasible after the conductance is increased.*

Remark 3.5. *For constant-power loads we have that the constant power demand* $(P_{\mathrm{c}})_i$ *at a load i vanishes if and only if the injected current* $(\mathcal{I}_L)_i$ *at the load vanishes, since* $(V_L)_i > 0$ *and*

$$(P_{\mathrm{c}})_i = -(P_L)_i = -(V_L)_i(\mathcal{I}_L)_i.$$

When the injected current at a load node vanishes, the load can be (algebraically) eliminated from the network through Kron reduction (e.g., see [24]), which results in a power grid of lower dimension. We let i be such a load and let j be a node for which the power demand is nonzero and increase the conductance G_{ij} *of the line between the nodes. In contrast to Theorem 3.2, the maximizing power demand* (6) *in the* Kron reduced *power grid could both increase or decrease when we increase* G_{ij}. *Indeed, it is observed in Figure 2 that if* $P_{\mathrm{c},2} = 0$, *then the maximal feasible value of* $P_{\mathrm{c},1}$ *decreases from 2.1 to 2 when* G_{12} *is increased from* 0.3 *to* 1. *Conversely, if we take* $P_{\mathrm{c},1} = 0$ *then the maximal feasible value of* $P_{\mathrm{c},2}$ *increases as* G_{12} *is increased.*

Theorem 3.3 implies that Braess' paradox may occur when multiple loads are present, as illustrated by Example 3.4. In fact, Braess' paradox may also occur for power grids with a single load and multiple sources— see [18, Supplementary Example 1].

We continue this section with a corollary to Theorem 3.3, which states that Braess' paradox can also occur if we only consider increasing the conductance of an existing line, and do not add new lines.

Corollary 3.6. *Consider a power grid with* Y *and* $V_S > \mathbb{0}$ *given and with at least two load nodes and at least two sources. If not all open-circuit voltages* (2) *are equal (i.e.,* $V_L^* \notin \operatorname{span}\{\mathbb{1}\}$*), then there exist a vectors of power demand* P_{c} *and a line between two load nodes such that the power flow is feasible and becomes unfeasible after increasing the conductance of the line.*

Proof. Suppose that for all lines between two load nodes i and j we have $(V_L^*)_i = (V_L^*)_j$. The load nodes in the power grid form a connected graph by assumption, which means that $(V_L^*)_i = (V_L^*)_j$ for all i, j, and hence $V_L^* \in \operatorname{span}\{\mathbb{1}\}$. This contradicts our assumption on V_L^*. Hence there exists loads i and j which are connected by a line and satisfy $(V_L^*)_i \neq (V_L^*)_j$. The proof of Theorem 3.3 shows that Braess' paradox can occur by increasing the conductance of the line between loads i and j. □

We remark that Theorem 3.3 and Corollary 3.6 extend naturally to DC power grids with resistive shunts (*e.g.*, see [24]). Indeed, resistive shunts can be thought of as lines connected to a source with a voltage potential that equals zero in the limit.

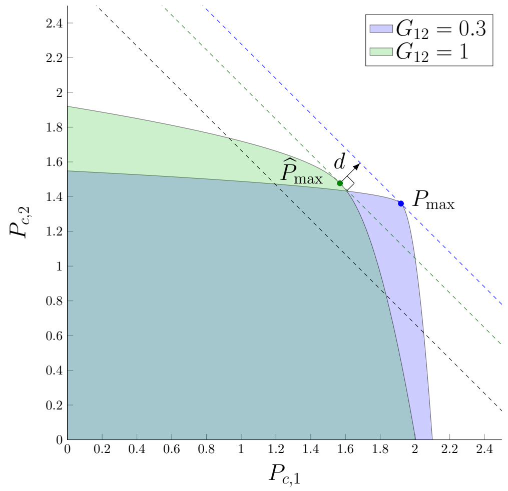

Figure 2: Plots for the sets of constant power demands P_c for which the power flow of the power grid in Example 3.4 is feasible. Throughout this paper the plots of such sets have been obtained through the parametrization of the feasibility boundary presented in [21]. The blue and green regions correspond to respectively $G_{12} = 0.3$ and $G_{12} = 1$. This corresponds to increasing the conductance G_{12} of the line between node 1 and node 2 by 0.7 (see Figure 1). The blue and green dashed lines are the points for which equality holds in (7) for the respective power grids. The decrease of the maximal total power demand is indicated by d. We observe that $P_{\max}$ is no longer feasible after the conductance is increased. The black dashed line corresponds the points for which equality in (9) holds. Theorem 3.3 shows that Braess' paradox may occur for all feasible vectors of constant power demands beyond the black dashed line, such as for $P_{\max}$.

3.2. Prevalence of Braess' paradox for power flow feasibility

In [2] it was shown that Braess' paradox is a widespread phenomenon which can occur in most traffic flow networks. Similarly, [25] showed that Braess' paradox may occur in any coupled-oscillator network such as AC power grids. In this section we validate that Braess' paradox is also common for DC power flow feasibility, and can occur in most practical DC power grids with constant-power loads, regardless of the grid topology.

Corollary 3.6 states that Braess' paradox may occur in the case where not all open-circuit voltages are equal. Due to the relation (2) between the source voltages and the open-circuit voltages, we may restate the theorem in terms of the source voltages.

Theorem 3.7. *Consider a power grid with Y and $V_S > \mathbb{0}$ given and with at least two load nodes and at least two sources. If Y_{LS} has full column rank and the source voltages are not all equal (i.e., $V_S \notin \operatorname{span}\{\mathbb{1}\}$), then there exist a vector of power demands P_c and a line between two load nodes such that the power flow is feasible and becomes unfeasible after increasing the line conductance.*

Proof. Note from (1) that $Y\mathbb{1} = \mathbb{0}$, which implies that

$$Y_{LL}\mathbb{1} + Y_{LS}\mathbb{1} = \mathbb{0}.$$

Rewriting this expression yields $\mathbb{1} = -{Y_{LL}}^{-1}Y_{LS}\mathbb{1}$. The matrix $-{Y_{LL}}^{-1}Y_{LS}$ has full column rank, and so by (2) we have $V_L^* = c\mathbb{1}$ for some $c > \mathbb{0}$ if and only if $V_S = c\mathbb{1}$. Corollary 3.6 implies the result. □

Theorem 3.7 shows that Braess' paradox may occur if Y_{LS} has full column rank and not all source voltages are equal. Apart from the condition on Y_{LS}, this statement does not depend on the topology of the grid. Hence, Braess' paradox may occur in both radial (tree) and meshed grid topologies.

The condition that Y_{LS} has full column rank can be interpreted as the property that there are no redundant voltage sources, in the sense that the removal of one source cannot be compensated by other sources. A more precise formulation and proof for this interpretation is given in Lemma A.2. To illustrate, Y_{LS} has full column rank if each load shares a line with at most one source and each source shares a line with at least one load. Conversely, if three sources share lines with only two loads, then (at least) one source is redundant. A study on benchmark power grids was performed to illustrate the prevalence of the condition that Y_{LS} has full-rank in practical power grids. The results are collected in Figure 4.

For all power grids with a single source node we have that Y_{LS} has full column rank. This shows that the premises of Theorem 3.3, Corollary 3.6 and Theorem 3.7 do not hold for power grids with a single source. However, since practical power grids commonly have multiple sources[1], and all source voltages are not likely to be the same, we conclude that Braess' paradox for power flow feasibility may occur in most practical DC power grids.

A natural follow-up question is if Braess' paradox can also occur if all source voltages are equal, for example in the case of a single source. When all voltage potentials at the sources are equal, we may replace all sources by a single source, which results in an equivalent power flow feasibility problem. Hence, to prove that Braess' paradox cannot occur when all source voltages are equal, studying the case of a single source node would be sufficient. The following example suggests that Braess' paradox does not occur for such power grids. A formal proof of this claim is lacking, and thus this question remains open.

Example 3.8. *Consider again the power grid with two loads and two sources, as depicted in Figure 1, where now $V_S = \begin{pmatrix} 2 & 2 \end{pmatrix}^\top$, and as in Example 3.4 we have $G_{12} = 0.3$, $G_{14} = 1$, $G_{23} = 5$ and $G_{34} = 1$. The corresponding open-circuit voltages are $V_L^* = \frac{2.24}{1.1}\begin{pmatrix} 1 & 1 \end{pmatrix}^\top$, which is a multiple of $\mathbb{1}$. The blue area in Figure 2 depicts the corresponding the set of all power demands P_c such that the power flow is feasible. We increase the conductance of this line by 0.7 and 1.7, which results in respectively the green and yellow areas in Figure 3, corresponding to the P_c such that the power flow is feasible. We observe that the blue area is contained in the green area, which in turn is contained in the yellow area. This suggests that power flow feasibility is not compromised when G_{ij} is increased. We also observe that $P_{\max}$ does not change when the conductance is increased, which follows from (10) in the proof of Theorem 3.3, since now*

[1]With the possible exception of power distribution grids with constant-power loads, which are commonly modeled with a single source or slack bus.

$(V_L^*)_1 = (V_L^*)_2$. *Note that taking $G_{12} \to \infty$ in the power grid in Figure 1 corresponds to shorting the line between nodes i and j. This corresponds to combining the two loads into a single constant-power load, where the constant power demand of the new load is the sum of the demands of the two loads. Since $n = 2$, this sum is bounded from above by $\mathbb{1}^\top P_{\max}$ by Proposition 2.2. Figure 3 suggests that the set of feasible power demands converges to the half-space described by* (7)*, which has the dashed line as boundary. Moreover, it suggests that Braess' paradox does not occur for this power grid when the source voltages are equal.*

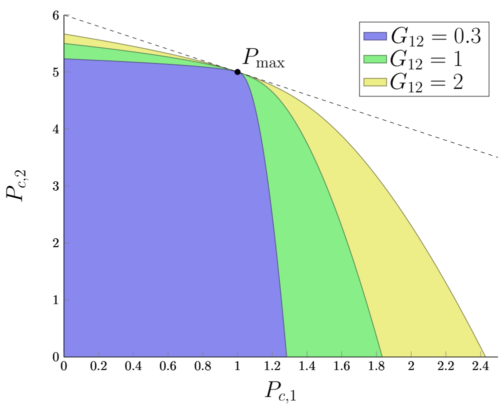

Figure 3: Plots for the set of feasible power demands for the power grid in Example 3.8. The blue, green and yellow regions correspond to respectively $G_{12} = 0.3$, $G_{12} = 1$ and $G_{12} = 2$. We observe that $P_{\max}$ is the same for all G_{12}.

Remark 3.9. *The examples in this paper are purely academic. The voltage potentials in the presented examples differ by a factor of three, which is helpful for the illustration of the phenomenon, but is extreme when compared to realistic power grids that operate at high voltage. In the case where the voltage potentials of the source voltages differ only slightly, the effect of Braess' paradox might be less pronounced. Indeed, in such a case the open-circuit voltages* (2) *also differ only slightly and hence the final term in* (9) *is small. The further study of Braess' paradox for power flow feasibility and its significance in the context of realistic power grids is an interesting research topic, but lies beyond the scope of this paper.*

4. Bounded uncertainties in line conductances

Section 3 argued that Theorem 3.7 applies to most practical power grids, which shows that Braess' paradox is a fundamental problem for power flow feasibility when line conductances are not precisely known. Indeed, Example 3.4 illustrates that the line conductances can influence the power flow feasibility, especially when a power grid operates close to its feasibility limits. The existence of Braess' paradox for power flow feasibility shows that exact measurements of line conductances are desirable for guaranteeing the safe operation of a power grid. Unfortunately, such measurements are often not available, and only rough estimates of line conductances may be known. This motivates us to study DC power grids with constant-power loads

MATPOWER 7.0 case name	Thm. 3.7 applies?	MATPOWER 7.0 case name	Thm. 3.7 applies?
case118	Yes	case6495rte	Yes
case1354pegase	Yes	case6515rte	Yes
case13659pegase	Yes	case6ww	Yes
case14	Yes	case9	Yes
case2383wp	Yes	case9241pegase	Yes
case24_ieee_rts	Yes	case9Q	Yes
case2736sp	Yes	case9target	Yes
case2737sop	Yes	case_ACTIVSg10k	Yes
case2746wop	Yes	case_ACTIVSg25k	Yes
case2746wp	Yes	case_ACTIVSg70k	Yes
case2848rte	Yes	case_ieee30	Yes
case2868rte	Yes	case_RTS_GMLC	Yes
case2869pegase	Yes	case_SyntheticUSA	Yes
case30	Yes		
case300	Yes	case145	No
case3012wp	Yes	case1888rte	No
case30pwl	Yes	case1951rte	No
case30Q	Yes	case4_dist	No
case3120sp	Yes	case4gs	No
case3375wp	Yes	case5	No
case39	Yes	case89pegase	No
case57	Yes	case_ACTIVSg200	No
case6468rte	Yes	case_ACTIVSg2000	No
case6470rte	Yes	case_ACTIVSg500	No

Figure 4: A table of cases from MATPOWER 7.0 [26, 27]. The MATPOWER cases were interpreted as DC power grids by studying the decoupled reactive power flow in a lossless setting, in accordance with [18]. In this analogy, PV buses and PQ buses in a MATPOWER case correspond to voltage sources and constant-power loads in a DC power grid, respectively. Cases that do not contain PV buses were omitted. The inductance of the lines in the MATPOWER cases were taken as the line conductances in the DC power grids. Recall that the current paper assumes that the graph induced by the loads is connected, and so each connected component induced by the loads and their neighboring sources within these DC power grids was analyzed separately. The second and fourth column in the table indicate whether or not Theorem 3.7 applies to one of these connected components, in which case there exists a feasible vector P_c of constant power demands and a line between two loads (PQ buses) for which increasing the conductance of the line leads to unfeasibility of the power flow, showing that Braess' paradox may occur. The Julia code that produce these results is found at [28]. Under the assumption that the obtained DC power grids are representative, the table implies that Braess' paradox may occur in the majority of the cases, which suggests that it is a prevalent phenomenon.

such that power flow feasibility is resilient to inaccuracies or deviations of the line conductances.

In the remainder of this paper we assume that no exact measurements of the line conductances are available. Instead, we assume that upper and lower bounds on the line conductances are known, which are represented by $\overline{G}_{ij} \geq 0$ and $\underline{G}_{ij} \geq 0$, respectively, and satisfy $\underline{G}_{ij} \leq \overline{G}_{ij}$. Note that the case where $\underline{G}_{ij} = 0$ and $\overline{G}_{ij} > 0$ corresponds to the possibility that nodes i and j are possibly not connected by a line.

Throughout this section we assume that $V_S > \mathbb{0}$ is known and fixed. We also assume that the conductance in at least one line between a load and any other node is not exactly known, which means $\underline{G}_{ij} \neq \overline{G}_{ij}$ some pair of nodes (i, j) where i or j is a load. We define the following notions.

Definition 4.1. *For a given Kirchhoff matrix Y we let $\mathcal{F}_Y$ denote*

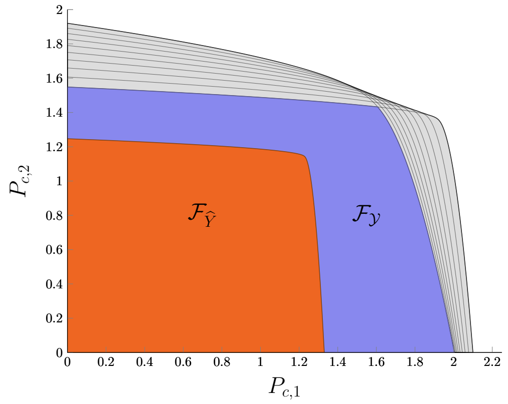

Figure 5: Plots of constant power demands for the power grid in Example 3.4. This means that Kirchhoff matrices $Y \in \mathcal{Y}$ are described by the bounds $\underline{G}_{12} = 0.3$, $\overline{G}_{12} = 1$, $\underline{G}_{14} = \overline{G}_{14} = 1$, $\underline{G}_{23} = \overline{G}_{23} = 5$ and $\underline{G}_{34} = \overline{G}_{34} = 1$. The gray regions depicts the power demands such that the power flow is feasible for different choices of line conductances. The blue region corresponds to the set $\mathcal{F}_{\mathcal{Y}}$ of power demands such that the power flow is strongly feasible, and is the intersection of all gray regions. The red region corresponds to the sufficient condition for strong feasibility in Theorem 4.4, and is contained in the blue region.

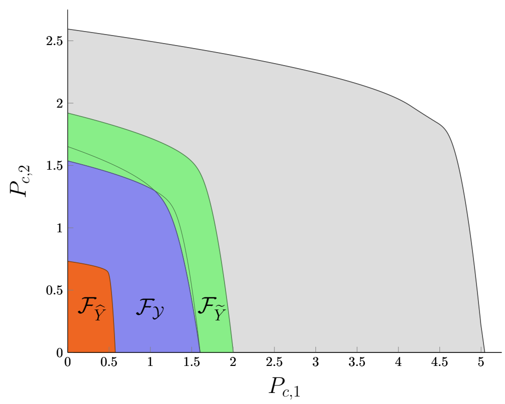

Figure 6: Plots of constant power demands for the power grid in Figure 1 with $V_S = (1 \quad 3)^\top$. The green region corresponds to the power demands such that the power grid with line conductances $\widetilde{G}_{12} = 1$, $\widetilde{G}_{14} = 1$, $\widetilde{G}_{23} = 5$ and $\widetilde{G}_{14} = 1$ has feasible power flow, similar to Example 3.4. The gray region describes the union of same sets when the line conductances deviate by at most 20% In this case we have $\underline{G}_{ij} = 0.8\,\widetilde{G}_{ij}$ and $\overline{G}_{ij} = 1.2\,\widetilde{G}_{ij}$ for all i, j. The blue region corresponds to the power demands such that the power grid is strongly feasible under the line uncertainties. The red region depicts the sufficient condition of Theorem 4.4, and is contained in the blue region.

the set of all constant power demands $P_\mathrm{c} \in \mathbb{R}^n$ such that the power flow is feasible. I.e.,

$$\mathcal{F}_Y := \{\, P_c \mid \exists V_L > \mathbb{0} : P_c = -[V_L](Y_{LL}V_L + Y_{LS}V_S)\,\}\,.$$

The set $\mathcal{F}_Y$ is closed and convex [20, Thm. 3.18], and can be constructed by parametrizing their boundary (see Thm. 3.8 and Thm 3.12 in [21]). Note that $\mathcal{F}_Y$ is independent of the principal submatrix Y_{SS}.

Definition 4.2. *We let $\mathcal{Y} \subseteq \mathbb{R}^{(n+m)\times(n+m)}$ denote the set of all Kirchhoff matrices Y defined in* (1) *that satisfy $\underline{G}_{ij} \le G_{ij} \le \overline{G}_{ij}$.*

Definition 4.3. *Consider a power grid with line uncertainties described by $\mathcal{Y}$ and with constant power demands $P_\mathrm{c} \in \mathbb{R}^n$. We say that the power grid is* strongly feasible *if the power flow is feasible for all Kirchhoff matrices $Y \in \mathcal{Y}$, or equivalently, if P_c is contained in the set*

$$\mathcal{F}_{\mathcal{Y}} := \bigcap_{Y\in\mathcal{Y}} \mathcal{F}_Y. \tag{11}$$

We remark that $\mathcal{F}_{\mathcal{Y}}$ is closed convex, since it is the intersection of closed convex sets. Because the set $\mathcal{F}_Y$ can be constructed, we can be numerically compute $\mathcal{F}_{\mathcal{Y}}$ for examples with few uncertainties, by explicitly computing the intersection of the sets $\mathcal{F}_Y$. This method is used to obtain Figure 5, which depicts the set of constant power demands for which the power grid described in Example 3.4 is strongly feasible. Note however that this explicit method is computationally intensive when many line conductances are uncertain. Hence a more tractable method to check is therefore desirable. An example with multiple uncertainties is found in Figure 6, where we consider the power grid in Example 3.4 with $G_{12} = 1$ and consider an uncertainty of 20%

4.1. A sufficient condition for strong feasibility

We conclude this section by presenting a sufficient condition for the strong feasibility of a power grid with line uncertainties. Our methodology relies on constructing an auxiliary power grid that features resistive shunts at the load nodes, which are lines that connect a node to ground. Adding resistive shunts to a power grid makes the power flow feasibility of a power grid more conservative, which we show in Lemma A.3. By taking an appropriate choice of the line conductances and resistive shunts, we prove that power flow feasibility of the auxiliary power grid implies that the power flow of the power grid with line uncertainties is strongly feasible.

To this end we introduce the following auxiliary power grid, based on the bounds $\underline{G}_{ij}$ and $\overline{G}_{ij}$, with same number of nodes and the same source voltages, but different line conductances. In this power grid we take $\underline{G}_{ij}$ as the conductance of the line between nodes i and j. In addition we consider resistive shunts at the nodes, which are lines that connect a node to ground. The line conductance of each shunt equals

$$G_{\mathrm{shunt},i} := \sum_k (\overline{G}_{ki} - \underline{G}_{ki}).$$

Since $\underline{G}_{ij} \neq \overline{G}_{ij}$ for some load node i, we know that at $G_{\mathrm{shunt},i} > 0$ for at last one load node i. The shunted Kirchhoff matrix $\widehat{Y} \in \mathbb{R}^{(n+m)\times(n+m)}$ (*e.g.*, see [24]) associated to this auxiliary power grid is given by

$$\widehat{Y}_{ij} := \begin{cases} \sum_k \underline{G}_{ki} + G_{\mathrm{shunt},i} = \sum_k \overline{G}_{ki} & \text{if } i = j \\ -\underline{G}_{ij} & \text{if } i \neq j \end{cases}. \tag{12}$$

Theorem 4.4. *If a vector $P_\mathrm{c} \in \mathbb{R}^n$ of constant power demands is such that the auxiliary power grid with shunted Kirchhoff matrix $\widetilde{Y}$ has feasible power flow, then any power grid with Kirchhoff matrix $Y \in \mathcal{Y}$ also has feasible power flow, implying that* *the power grid is strongly feasible for this P_c. Put differently, we have $\mathcal{F}_{\widehat{Y}} \subseteq \mathcal{F}_{\mathcal{Y}}$. Moreover, P_c never lies on the boundary of $\mathcal{F}_Y$ for any $Y \in \mathcal{Y}$.*

Proof. Let $P_c \in \mathcal{F}_{\widehat{Y}}$, which means that there exists $\widetilde{V}_L > \mathbb{0}$ such that

$$P_c = -[\widetilde{V}_L](\widehat{Y}_{LL}\widetilde{V}_L + \widehat{Y}_{LS} V_S).$$

Let $Y \in \mathcal{Y}$. The matrix $\widehat{Y}$ satisfies $\widehat{Y} - Y \geq 0$ since $\underline{G}_{ij} \leq G_{ij} \leq \overline{G}_{ij}$. This means that for all $V_L > \mathbb{0}$ we have

$$(\widehat{Y}_{LL} - Y_{LL})V_L + (\widehat{Y}_{LS} - Y_{LS})V_S \geq \mathbb{0}. \tag{13}$$

Since $\underline{G}_{ij} \neq \overline{G}_{ij}$ for some load node i we have $\widehat{Y}_{LL} \neq Y_{LL}$ or $\widehat{Y}_{LS} \neq Y_{LS}$, and hence we never have equality in (13). Using (13) we observe that

$$\begin{aligned} P_c &= -[\widetilde{V}_L](\widehat{Y}_{LL}\widetilde{V}_L + \widehat{Y}_{LS} V_S) \\ &\lneqq -[\widetilde{V}_L](Y_{LL}\widetilde{V}_L + Y_{LS} V_S) =: \widetilde{P}_c. \end{aligned}$$

By definition of $\mathcal{F}_Y$ we have that $\widetilde{P}_c \in \mathcal{F}_Y$. Since $P_c \lneqq \widetilde{P}_c$ it follows from [21, Lem. 4.1] that $P_c \in \operatorname{int}(\mathcal{F}_Y)$, the interior of $\mathcal{F}_Y$. Hence, $\mathcal{F}_{\widehat{Y}} \subseteq \operatorname{int}(\mathcal{F}_Y)$ for all $Y \in \mathcal{Y}$, and thus $\mathcal{F}_{\widehat{Y}} \subseteq \mathcal{F}_{\mathcal{Y}}$. □

Theorem 4.4 presents a sufficient condition for the strong feasibility of a power grid, in terms of the inclusion $P_c \in \mathcal{F}_{\widehat{Y}}$. There exists effective methods to verify this inclusion. It was shown in [20, Thm. 3.22] that the condition $P_c \in \mathcal{F}_{\widehat{Y}}$ is equivalent to solving a linear matrix inequality. This condition may also be verified using the method proposed in [19]. Alternatively, if P_c is nonnegative then this condition can be verified using the scalar-valued inequality in [21, Thm. 3.15].

The fact that any $P_c \in \mathcal{F}_{\widehat{Y}}$ never lies on the boundary of $\mathcal{F}_Y$ indicates that this condition is not tight and therefore conservative. This is verified in Figures 5 and 6, in which the set $\mathcal{F}_{\widehat{Y}}$ does not touch the boundary of $\mathcal{F}_{\mathcal{Y}}$. This conservativeness can be attributed to the resistive shunts in the auxiliary power grid, since resistive shunts make power flow feasibility more conservative by Lemma A.3, and since shunts are not present in the power grid described by $\mathcal{Y}$.

The auxiliary power grid is not unique to the power grid described by $\mathcal{Y}$. This is observed by comparing Figures 7 and 5, which consider two power grids with different uncertainties in the lines, but where the sets $\mathcal{F}_{\widehat{Y}}$ are equal for both cases. Note however that the sets $\mathcal{F}_{\mathcal{Y}}$ in these figures are not equal, which shows that, loosely speaking, the set $\mathcal{F}_{\widehat{Y}}$ is not an accurate approximation of $\mathcal{F}_{\mathcal{Y}}$, which further illustrates the conservativeness of Theorem 4.4.

5. Conclusion and discussion

In this paper we have shown that an analogue of Braess' paradox may occur in the power flow feasibility of a DC power grid. The observed phenomenon states that an increase of a line conductance has the potential to destroy the feasibility of the power flow in a power grid. We have shown that this phenomenon may occur in most practical power grids with multiple sources. This has motivated the study of power grids where the exact line conductances are unknown, and only upper and lower bounds are available. We introduced the concept of strong feasibility, which asks when power flow is feasible for all line conductances within these bounds. We concluded by presenting a sufficient condition for the strong feasibility of the power flow in such power grids.

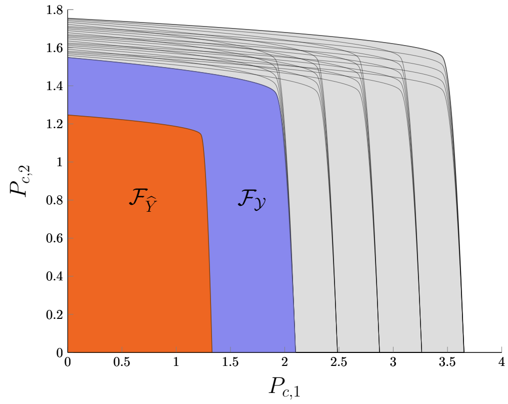

Figure 7: Plots of constant power demands for the power grid in Figure 1 with $V_S = (1 \quad 3)^\top$ and with bounded line conductances. We have $\underline{G}_{12} = \overline{G}_{12} = 0.3$, $\underline{G}_{14} = 1$, $\overline{G}_{14} = 1.7$, $\underline{G}_{23} = 5$, $\overline{G}_{23} = 5.7$ and $\underline{G}_{34} = \overline{G}_{34} = 1$. The gray regions depicts the power demands such that the power flow is feasible for different choices of line conductances. The blue region corresponds to the set $\mathcal{F}_{\mathcal{Y}}$ of power demands such that the power flow is strongly feasible, and is the intersection of all gray regions. The red region corresponds to the sufficient condition for strong feasibility in Theorem 4.4, and is contained in the blue region. We observe that the set $\mathcal{F}_{\widehat{Y}}$ coincides with the set $\mathcal{F}_{\widehat{Y}}$ in Figure 5. This follows since the matrices $\widehat{Y}$ are the equal, with the exception of the Y_{SS} submatrix.

Further directions of research are as follows. We have suggested to combat Braess' paradox by considering bounded line conductances and introducing the notion of strong feasibility. However, there may be alternative methods to avoid Braess' paradox that are worth studying, such as a probabilistic approach. Furthermore, many question regarding strong feasibility can be studied. For example, is the intersection in (11) equivalent to a finite intersection? Or more ambitiously, can we formulate a necessary and sufficient condition for a power grid with bounded line conductances to have strongly feasible power flow? Finally, what are the implications of the presented results in the context of traffic networks, or other flow networks?

Acknowledgment

The author would like to thank Claudio De Persis and Arjan van der Schaft for their valuable suggestions and discussions.

Appendix A. Appendix

Lemma A.1. *Let $A = A^\top \in \mathbb{R}^{n\times n}$ be nonsingular, let $b, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If $1 + cb^\top A^{-1} b \neq 0$ then*

$$v^\top (A + cbb^\top)^{-1} v = v^\top A^{-1} v - c\frac{(v^\top A^{-1} b)^2}{1 + cb^\top A^{-1} b}.$$

Proof. By the Sherman-Morrison formula [29] we have that $(A + cbb^\top)^{-1}$ exists and satisfies

$$(A + cbb^\top)^{-1} = A^{-1} - c(1 + cb^\top A^{-1}b)^{-1}A^{-1}bb^\top A^{-1}.$$

The result is obtained through multiplication by $v^\top$ and v, and noting that $A^{-\top} = A^{-1}$. □

Lemma A.2. *The matrix $Y_{LS} \in \mathbb{R}^{n\times m}$ has full column rank if and only if for each $V_S > \mathbb{0}$ there does not exists a vector $\widehat{V}_S \geq \mathbb{0}$ such that $Y_{LS}V_S = Y_{LS}\widehat{V}_S$ and $(\widehat{V}_S)_i = 0$ for some i (i.e., none of the sources are redundant).*

Proof. (⇒): If $Y_{LS}V_S = Y_{LS}\widehat{V}_S$ and $(\widehat{V}_S)_i = 0 < (V_S)_i$, then $\mathbb{0} \neq V_S - \widehat{V}_S \in \ker Y_{LS}$ and so Y_{LS} does not have full column rank.

(⇐): Suppose $Y_{LS} \in \mathbb{R}^{n\times m}$ does not have full column rank. Let w be a nonzero vector in the kernel of Y_{LS}. Without loss of generality we assume that w has a negative entry. It follows that for some $\theta > 0$ we have $\widehat{V}_S := V_S + \theta w \geq \mathbb{0}$ with equality for at least one entry. But this means that $Y_{LS}V_S = Y_{LS}\widehat{V}_S$ and $(\widehat{V}_S)_i = 0$ for some i, which contradicts our premise. Hence $Y_{LS} \in \mathbb{R}^{n\times m}$ has full column rank. □

Lemma A.3. *Consider a power grid with Kirchhoff matrix Y. Adding resistive shunts to the load nodes makes power flow feasibility more conservative. More explicitly, let D be a nonnegative diagonal matrix with D_{ii} positive for some load node i. Let P_c and $V_S > \mathbb{0}$ be given. If a power grid with Kirchhoff matrix $Y + D$ has feasible power flow (i.e., $P_c \in \mathcal{F}_{Y+D}$), then a power grid with Kirchhoff matrix Y has feasible power flow (i.e., $P_c \in \mathcal{F}_Y$) and P_c does not lie on the boundary of $\mathcal{F}_Y$.*

Proof. Let $P_c \in \mathcal{F}_{Y+D}$, which means that there exists $\widetilde{V}_L > \mathbb{0}$ such that

$$P_c = -[\widetilde{V}_L]((Y + D)_{LL}\widetilde{V}_L + (Y + D)_{LS}V_S).$$

For all $V_L > \mathbb{0}$ we have $D_{LL}V_L \gneq \mathbb{0}$, where equality does not hold since D_{ii} positive for some load node i. Note also that $D_{LS} = 0$ since D is diagonal. We therefore have

$$\begin{aligned} P_c &= -[\widetilde{V}_L](\widehat{Y}_{LL}\widetilde{V}_L + \widehat{Y}_{LS}V_S) \\ &\lneq -[\widetilde{V}_L](Y_{LL}\widetilde{V}_L + Y_{LS}V_S) =: \widetilde{P}_c. \end{aligned}$$

By definition of $\mathcal{F}_Y$ we have that $\widetilde{P}_c \in \mathcal{F}_Y$. Since $P_c \lneq \widetilde{P}_c$, [21, Lem. 4.1] implies that P_c lies in the interior of $\mathcal{F}_Y$. □

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